4. Let V be the set of all pairs (x, y) of real numbers, and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, y + y_1)$$
$$c(x, y) = (cx, y).$$

Is V, with these operations, a vector space over the field of real numbers?

5. On R^n, define two operations

$$\alpha \oplus \beta = \alpha - \beta$$
$$c \cdot \alpha = -c\alpha.$$

The operations on the right are the usual ones. Which of the axioms for a vector space are satisfied by $(R^n, \oplus, \cdot)$?

6. Let V be the set of all complex-valued functions f on the real line such that (for all t in R)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$(f + g)(t) = f(t) + g(t)$$
$$(cf)(t) = cf(t)$$

is a vector space over the field of *real* numbers. Give an example of a function in V which is not real-valued.

7. Let V be the set of pairs (x, y) of real numbers and let F be the field of real numbers. Define

$$(x, y) + (x_1, y_1) = (x + x_1, 0)$$
$$c(x, y) = (cx, 0).$$

Is V, with these operations, a vector space?

2.2. Subspaces

In this section we shall introduce some of the basic concepts in the study of vector spaces.

Definition. *Let* V *be a vector space over the field* F. *A* **subspace** *of* V *is a subset* W *of* V *which is itself a vector space over* F *with the operations of vector addition and scalar multiplication on* V.

A direct check of the axioms for a vector space shows that the subset W of V is a subspace if for each α and β in W the vector $\alpha + \beta$ is again in W; the 0 vector is in W; for each α in W the vector $(-\alpha)$ is in W; for each α in W and each scalar c the vector $c\alpha$ is in W. The commutativity and associativity of vector addition, and the properties (4)(a), (b), (c), and (d) of scalar multiplication do not need to be checked, since these are properties of the operations on V. One can simplify things still further.

Theorem 1. *A non-empty subset* W *of* V *is a subspace of* V *if and only if for each pair of vectors* α, β *in* W *and each scalar* c *in* F *the vector* $c\alpha + \beta$ *is again in* W.

Proof. Suppose that W is a non-empty subset of V such that $c\alpha + \beta$ belongs to W for all vectors α, β in W and all scalars c in F. Since W is non-empty, there is a vector ρ in W, and hence $(-1)\rho + \rho = 0$ is in W. Then if α is any vector in W and c any scalar, the vector $c\alpha = c\alpha + 0$ is in W. In particular, $(-1)\alpha = -\alpha$ is in W. Finally, if α and β are in W, then $\alpha + \beta = 1\alpha + \beta$ is in W. Thus W is a subspace of V.

Conversely, if W is a subspace of V, α and β are in W, and c is a scalar, certainly $c\alpha + \beta$ is in W. ∎

Some people prefer to use the $c\alpha + \beta$ property in Theorem 1 as the definition of a subspace. It makes little difference. The important point is that, if W is a non-empty subset of V such that $c\alpha + \beta$ is in V for all α, β in W and all c in F, then (with the operations inherited from V) W is a vector space. This provides us with many new examples of vector spaces.

EXAMPLE 6.

(a) If V is any vector space, V is a subspace of V; the subset consisting of the zero vector alone is a subspace of V, called the **zero subspace** of V.

(b) In F^n, the set of n-tuples $(x_1, \ldots, x_n)$ with $x_1 = 0$ is a subspace; however, the set of n-tuples with $x_1 = 1 + x_2$ is not a subspace $(n \geq 2)$.

(c) The space of polynomial functions over the field F is a subspace of the space of all functions from F into F.

(d) An $n \times n$ (square) matrix A over the field F is **symmetric** if $A_{ij} = A_{ji}$ for each i and j. The symmetric matrices form a subspace of the space of all $n \times n$ matrices over F.

(e) An $n \times n$ (square) matrix A over the field C of complex numbers is **Hermitian** (or **self-adjoint**) if

$$A_{jk} = \overline{A_{kj}}$$

for each j, k, the bar denoting complex conjugation. A 2×2 matrix is Hermitian if and only if it has the form

$$\begin{bmatrix} z & x + iy \\ x - iy & w \end{bmatrix}$$

where x, y, z, and w are real numbers. The set of all Hermitian matrices is *not* a subspace of the space of all $n \times n$ matrices over C. For if A is Hermitian, its diagonal entries $A_{11}, A_{22}, \ldots$, are all real numbers, but the diagonal entries of iA are in general not real. On the other hand, it is easily verified that the set of $n \times n$ complex Hermitian matrices is a vector space over the field R of real numbers (with the usual operations).

EXAMPLE 7. **The solution space of a system of homogeneous linear equations.** Let A be an $m \times n$ matrix over F. Then the set of all $n \times 1$ (column) matrices X over F such that $AX = 0$ is a subspace of the space of all $n \times 1$ matrices over F. To prove this we must show that $A(cX + Y) = 0$ when $AX = 0$, $AY = 0$, and c is an arbitrary scalar in F. This follows immediately from the following general fact.

Lemma. *If* A *is an* m $\times$ n *matrix over* F *and* B, C *are* n $\times$ p *matrices over* F *then*

(2-11)   A(dB + C) = d(AB) + AC

for each scalar d *in* F.

Proof. $[A(dB + C)]_{ij} = \sum_k A_{ik}(dB + C)_{kj}$

$$= \sum_k (dA_{ik}B_{kj} + A_{ik}C_{kj})$$

$$= d \sum_k A_{ik}B_{kj} + \sum_k A_{ik}C_{kj}$$

$$= d(AB)_{ij} + (AC)_{ij}$$

$$= [d(AB) + AC]_{ij}. \quad \blacksquare$$

Similarly one can show that $(dB + C)A = d(BA) + CA$, if the matrix sums and products are defined.

Theorem 2. *Let* V *be a vector space over the field* F. *The intersection of any collection of subspaces of* V *is a subspace of* V.

Proof. Let $\{W_a\}$ be a collection of subspaces of V, and let $W = \bigcap_a W_a$ be their intersection. Recall that W is defined as the set of all elements belonging to every W_a (see Appendix). Since each W_a is a subspace, each contains the zero vector. Thus the zero vector is in the intersection W, and W is non-empty. Let α and β be vectors in W and let c be a scalar. By definition of W, both α and β belong to each W_a, and because each W_a is a subspace, the vector $(c\alpha + \beta)$ is in every W_a. Thus $(c\alpha + \beta)$ is again in W. By Theorem 1, W is a subspace of V. $\quad \blacksquare$

From Theorem 2 it follows that if S is any collection of vectors in V, then there is a smallest subspace of V which contains S, that is, a subspace which contains S and which is contained in every other subspace containing S.

Definition. *Let* S *be a set of vectors in a vector space* V. *The* **subspace spanned** *by* S *is defined to be the intersection* W *of all subspaces of* V *which contain* S. *When* S *is a finite set of vectors,* S $= \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, *we shall simply call* W *the* **subspace spanned by the vectors** $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Theorem 3. *The subspace spanned by a non-empty subset* S *of a vector space* V *is the set of all linear combinations of vectors in* S.

Proof. Let W be the subspace spanned by S. Then each linear combination

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$$

of vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ in S is clearly in W. Thus W contains the set L of all linear combinations of vectors in S. The set L, on the other hand, contains S and is non-empty. If α, β belong to L then α is a linear combination,

$$\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m$$

of vectors α_i in S, and β is a linear combination,

$$\beta = y_1\beta_1 + y_2\beta_2 + \cdots + y_n\beta_n$$

of vectors β_j in S. For each scalar c,

$$c\alpha + \beta = \sum_{i=1}^{m} (cx_i)\alpha_i + \sum_{j=1}^{n} y_j\beta_j.$$

Hence $c\alpha + \beta$ belongs to L. Thus L is a subspace of V.

Now we have shown that L is a subspace of V which contains S, and also that any subspace which contains S contains L. It follows that L is the intersection of all subspaces containing S, i.e., that L is the subspace spanned by the set S. ∎

Definition. *If* $S_1, S_2, \ldots, S_k$ *are subsets of a vector space* V, *the set of all sums*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

of vectors α_i *in* S_i *is called the* **sum** *of the subsets* $S_1, S_2, \ldots, S_k$ *and is denoted by*

$$S_1 + S_2 + \cdots + S_k$$

or by

$$\sum_{i=1}^{k} S_i.$$

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum

$$W = W_1 + W_2 + \cdots + W_k$$

is easily seen to be a subspace of V which contains each of the subspaces W_i. From this it follows, as in the proof of Theorem 3, that W is the subspace spanned by the union of $W_1, W_2, \ldots, W_k$.

EXAMPLE 8. Let F be a subfield of the field C of complex numbers. Suppose

$$\alpha_1 = (1, 2, 0, 3, 0)$$
$$\alpha_2 = (0, 0, 1, 4, 0)$$
$$\alpha_3 = (0, 0, 0, 0, 1).$$

By Theorem 3, a vector α is in the subspace W of F^5 spanned by α_1, α_2, α_3 if and only if there exist scalars c_1, c_2, c_3 in F such that

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3.$$

Thus W consists of all vectors of the form

$$\alpha = (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$$

where c_1, c_2, c_3 are arbitrary scalars in F. Alternatively, W can be described as the set of all 5-tuples

$$\alpha = (x_1, x_2, x_3, x_4, x_5)$$

with x_i in F such that

$$x_2 = 2x_1$$
$$x_4 = 3x_1 + 4x_3.$$

Thus $(-3, -6, 1, -5, 2)$ is in W, whereas $(2, 4, 6, 7, 8)$ is not.

EXAMPLE 9. Let F be a subfield of the field C of complex numbers, and let V be the vector space of all 2×2 matrices over F. Let W_1 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$$

where x, y, z are arbitrary scalars in F. Finally, let W_2 be the subset of V consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

where x and y are arbitrary scalars in F. Then W_1 and W_2 are subspaces of V. Also

$$V = W_1 + W_2$$

because

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}.$$

The subspace $W_1 \cap W_2$ consists of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}.$$

EXAMPLE 10. Let A be an $m \times n$ matrix over a field F. The **row vectors** of A are the vectors in F^n given by $\alpha_i = (A_{i1}, \ldots, A_{in})$, $i = 1, \ldots,$ m. The subspace of F^n spanned by the row vectors of A is called the **row**

space of A. The subspace considered in Example 8 is the row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is also the row space of the matrix

$$B = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -4 & -8 & 1 & -8 & 0 \end{bmatrix}.$$

EXAMPLE 11. Let V be the space of all polynomial functions over F. Let S be the subset of V consisting of the polynomial functions $f_0, f_1, f_2, \ldots$ defined by

$$f_n(x) = x^n, \qquad n = 0, 1, 2, \ldots .$$

Then V is the subspace spanned by the set S.

Exercises

1. Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n)$ in R^n are subspaces of R^n $(n \geq 3)$?

    (a) all α such that $a_1 \geq 0$;
    (b) all α such that $a_1 + 3a_2 = a_3$;
    (c) all α such that $a_2 = a_1^2$;
    (d) all α such that $a_1 a_2 = 0$;
    (e) all α such that a_2 is rational.

2. Let V be the (real) vector space of all functions f from R into R. Which of the following sets of functions are subspaces of V?

    (a) all f such that $f(x^2) = f(x)^2$;
    (b) all f such that $f(0) = f(1)$;
    (c) all f such that $f(3) = 1 + f(-5)$;
    (d) all f such that $f(-1) = 0$;
    (e) all f which are continuous.

3. Is the vector $(3, -1, 0, -1)$ in the subspace of R^5 spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$, and $(1, 1, 9, -5)$?

4. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in R^5 which satisfy

$$2x_1 - \; x_2 + \tfrac{4}{3}x_3 - \; x_4 \qquad\quad = 0$$

$$x_1 \qquad\quad + \tfrac{2}{3}x_3 \qquad\quad - \; x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0.$$

Find a finite set of vectors which spans W.

5. Let F be a field and let n be a positive integer $(n \geq 2)$. Let V be the vector space of all $n \times n$ matrices over F. Which of the following sets of matrices A in V are subspaces of V?

  (a) all invertible A;
  (b) all non-invertible A;
  (c) all A such that $AB = BA$, where B is some fixed matrix in V;
  (d) all A such that $A^2 = A$.

6. (a) Prove that the only subspaces of R^1 are R^1 and the zero subspace.

  (b) Prove that a subspace of R^2 is R^2, or the zero subspace, or consists of all scalar multiples of some fixed vector in R^2. (The last type of subspace is, intuitively, a straight line through the origin.)

  (c) Can you describe the subspaces of R^3?

7. Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other.

8. Let V be the vector space of all functions from R into R; let V_e be the subset of even functions, $f(-x) = f(x)$; let V_o be the subset of odd functions, $f(-x) = -f(x)$.

  (a) Prove that V_e and V_o are subspaces of V.
  (b) Prove that $V_e + V_o = V$.
  (c) Prove that $V_e \cap V_o = \{0\}$.

9. Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{0\}$. Prove that for each vector α in V there are *unique* vectors α_1 in W_1 and α_2 in W_2 such that $\alpha = \alpha_1 + \alpha_2$.

2.3. Bases and Dimension

We turn now to the task of assigning a dimension to certain vector spaces. Although we usually associate 'dimension' with something geometrical, we must find a suitable algebraic definition of the dimension of a vector space. This will be done through the concept of a basis for the space.

Definition. *Let* V *be a vector space over* F. *A subset* S *of* V *is said to be* **linearly dependent** (*or simply,* **dependent**) *if there exist distinct vectors* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *in* S *and scalars* $c_1, c_2, \ldots, c_n$ *in* F, *not all of which are* 0, *such that*

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = 0.$$

A set which is not linearly dependent is called **linearly independent.** *If the set* S *contains only finitely many vectors* $\alpha_1, \alpha_2, \ldots, \alpha_n$, *we sometimes say that* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *are dependent (or independent) instead of saying* S *is dependent (or independent).*