Linear Algebra

Determinant

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Introduction

Determinant of a matrix

The determinant of a 2×2 matrix $A = [a_{ij}]$ is the number: Why???

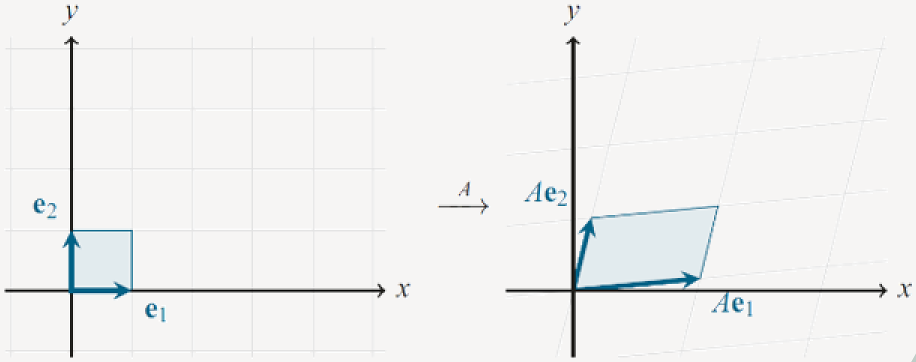
$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

- The absolute value of the determinant of a matrix measures how much it expands space when acting as a linear transformation. That is, it is the area (or volume, or hypervolume, depending on the dimension) of the output of the unit square, cube, or hypercube after it is acted upon by the matrix.

Geometric interpretation

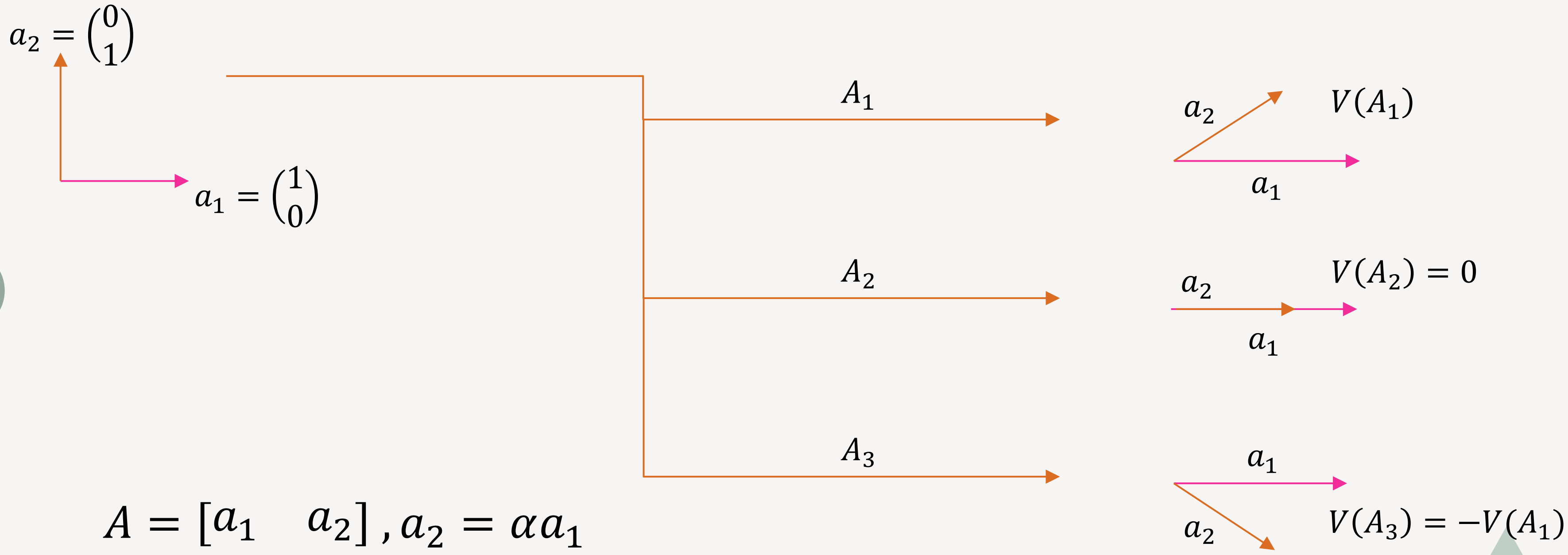
- The **volume** is a n-alternating multilinear map on all n-parallelepipeds such that the volume of standard unit parallelepiped is one.

$$\frac{\textit{volume of output region}}{\textit{volume of input region}}$$



A 2×2 matrix A stretches the unit square (with sides e_1 and e_2) into a parallelogram with sides Ae_1 and Ae_2 (the columns of A). The determinant of A is the area of this parallelogram.

Geometric interpretation



$$A = [a_1 \quad a_2]\,, a_2 = \alpha a_1$$

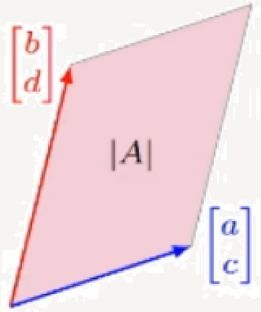
$$V(a_1, a_2) = -V(a_2, a_1)$$

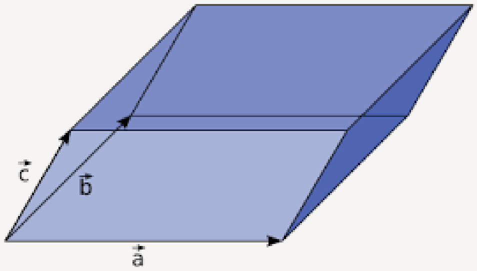
Determinants as Area or Volume

- If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is det(A)
- If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is det(A)
- Examples:

$$\text{Volume of } \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

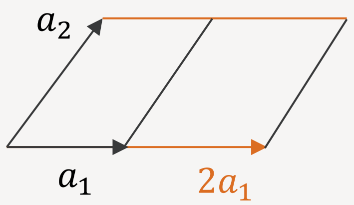
It is a rotation with $\boldsymbol{\theta}$ degree





Volume Example

- 2D Case
  - $V(a_1, a_2)$
  - $V(2a_1, a_2) = 2V(a_1, a_2)$ => $V(\beta a_2, a_1) = -\beta V(a_1, a_2)$
  - $V(-a_1, a_2) = -V(a_1, a_2)$



Volume

Definition

Every $\boldsymbol{n}$-dimensional parallelepiped with $\{\boldsymbol{a_1}, \dots, \boldsymbol{a_n}\}$ as legs is associated with a real number, called its volume which has the following properties:

If we stretch a parallelepiped by multiplying one of its legs by a scalar λ, its volume gets multiplied by λ.

If we add a vector ω to i-th legs of a n-dimensional parallelepiped with $\{a_1, \dots, a_i, a_{i+1}, \dots, a_n\}$, then its volume is the sum of the volume from $\{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n\}$ and the volume of $\{a_1, \dots, a_{i-1}, \omega, a_{i+1}, \dots, a_n\}$.

The volume changes sign when two legs are exchanged.

The volume of the parallelepiped with $\{e_1, \dots, e_n\}$ is one.

$$\phi : \underbrace{V \times \cdots \times V}_{n} \rightarrow \mathbb{R}$$

02

Linear Form

What are Linear Functions?

- $f: R^n \to R$ means that f is a function that maps real n-vectors to real numbers
- $f(x)$ is the value of function f at x (x is referred to as the argument of the function).
- $f(x) = f(x_1, x_2, \dots, x_n)$: where $x_1, x_2, \dots, x_n$ are arguments

Definition

A function $f: R^n \to R$ is linear if it satisfies the following two properties:

- **Additivity:** For any n-vector x and y, $f(x + y) = f(x) + f(y)$
- **Homogeneity:** For any n-vector x and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$

Superposition property:

Definition

Superposition property:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Note

- ❑ A function that satisfies the superposition property is called linear

What are Linear Functions?

- If a function f is linear, superposition extends to linear combinations of any number of vectors:

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

Inner product is Linear Function?

Theorem 1

A function defined as the inner product of its argument with some fixed vector is linear.

Proof?

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

What are Linear Functions?

Theorem 2

If a function is linear, then it can be expressed as the inner product of its argument with some fixed vector.

Proof?

What are Linear Functions?

Theorem 3

The representation of a linear function f in a specific basic as:
$f(x) = a^T x$ is unique, which means that there is only one vector a for which $f(x) = a^T x$ holds for all x.

Proof?

Linear Form Examples

Example

- Is average a linear function?
- Is maximum a linear function?

03

Bilinear Form

Bilinear Form over a real vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{F}$. Then a function $f: V \times W \ \rightarrow \ \mathbb{F}$ is called a **bilinear form** if it satisfies the following properties:

a) It is linear in its first argument:

i. $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and

ii. $f(c\mathbf{v_1}, \mathbf{w}) = cf(\mathbf{v_1}, \mathbf{w})$ for all $c \in \mathbb{F}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

b) It is linear in its second argument:

i. $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and

ii. $f(\mathbf{v}, c\mathbf{w_1}) = cf(\mathbf{v}, \mathbf{w_1})$ for all $c \in \mathbb{F}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

Bilinear Form

Note

Let V be a vector space over a field $\mathbb{F}$. Then the **dual** of V, denoted by V^*, is the vector space consisting of all linear forms on V.

Example

Let V be a vector space over a field $\mathbb{F}$. Show that the function $g: V^* \times V \rightarrow \mathbb{F}$ defined by

$$g(f, \mathrm{v}) = f(\mathrm{v}) \text{ for all } f \in V^*, \mathrm{v} \in V$$

is a bilinear form.

Symmetric Bilinear Form

Definition

A **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V is called symmetric if for all $v, w \in V$:

$$f(v, w) = f(w, v)$$

Bilinear Form arises from a matrix

Theorem 4

Every **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space V arises from a matrix for all $v, w \in V$:

$$f(v, w) = v^T A w$$

Proof?

Associated Matrices

Definition

If V is a finite-dimensional vector space, $B = \{b_1, \dots, b_n\}$ is a basis of V, and $f: V \times V \rightarrow \mathbb{F}$ be a **bilinear form** function the associated matrix A of f with respect to B is the matrix $[f]_B \in \mathbb{F}^{n\times n}$ whose (i, j)-entry is the value $f(b_i, b_j)$.

$$f(v, w) = v^T A w = v^T [f]_B w$$

$$[f]_{\mathcal{B}} = \begin{pmatrix} f(b_1, b_1) & \dots & f(b_1, b_n) \\ \vdots & & \vdots \\ f(b_n, b_1) & \dots & f(b_n, b_n) \end{pmatrix}$$

Associated Matrices

Note

The associated matrix changes if we use a different basis.

Example

For the bilinear form $f\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}c\\d\end{bmatrix}\right) = 2ac + 4ad - bc$ on $\mathbb{F}^2$, find $[f]_B$ for basis $B = \{\begin{bmatrix}2\\1\end{bmatrix},\begin{bmatrix}-1\\4\end{bmatrix}\}$ and $[f]_P$ for basis P= $\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\}$

04

Bilinear Form Over Complex Vector Space

Bilinear Form over a complex vector space

Definition

Suppose V and W are vector spaces over the same field $\mathbb{C}$. Then a function $\alpha: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

It is linear in its first argument:

$\alpha(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = \alpha(\mathbf{v_1}, \mathbf{w}) + \alpha(\mathbf{v_2}, \mathbf{w})$ and

$\alpha(\lambda \mathbf{v_1}, \mathbf{w}) = \lambda \alpha(\mathbf{v_1}, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

It is conjugate linear in its second argument:

$\alpha(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = \alpha(\mathbf{v}, \mathbf{w_1}) + \alpha(\mathbf{v}, \mathbf{w_2})$ and

$\alpha(\mathbf{v}, \lambda \mathbf{w_1}) = \bar{\lambda} \alpha(\mathbf{v}, \mathbf{w_1})$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

The set of bilinear forms on $\mathbf{v}$ is denoted by $\mathbf{v}^2$.

Bilinear Form over a complex vector space

Bilinear forms on $\mathbb{R}^n$	Bilinear forms on $\mathbb{C}^n$
Linear in the first variable	Linear in the first variable
Linear in the second variable	Conjugate linear in the second variable

05

Alternating bilinear form

Alternating bilinear form

Definition

A bilinear form $\alpha \in V^{(2)}$ is called alternating if

$$\alpha(v, v) = 0$$

for all $v \in V$. The set of alternating bilinear forms on V is denoted by $V_{alt}^{(2)}$.

Example

Suppose $\varphi, \tau \in V'$. Then the bilinear form α on V defined by is alternating.

$$\alpha(u, \omega) = \varphi(u)\tau(\omega) - \varphi(\omega)\tau(u)$$

Alternating bilinear form

Theorem (1)

A bilinear form α on V is alternating if and only if

$$\alpha(u,\omega) = -\,\alpha(\omega,u)$$

For all $u,\omega \in V$.

Proof

Alternating bilinear form

Theorem (2)

The sets $V_{sym}^{(2)}$ and $V_{alt}^{(2)}$ are subspaces of $V^{(2)}$. Furthermore,

$$V^{(2)} = V_{sym}^{(2)} \oplus V_{alt}^{(2)}$$

Proof

06

Multilinear Form

Multilinear Forms

Definition

Suppose $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_p$ are vector spaces over the same field $\mathbb{F}$. A function

$$f : \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_p \to \mathbb{F}$$

is called a multilinear form ($\mathrm{f} \in \mathrm{V}^{(\mathrm{p})}$) if, for each $1 \leq j \leq p$ and each $\mathrm{v}_1 \in \mathcal{V}_1, \mathrm{v}_2 \in \mathcal{V}_2, \ldots, \mathrm{v}_p \in \mathcal{V}_p$, it is the case that the function $g : \mathcal{V}_j \to \mathbb{F}$ defined by

$$g(\mathrm{v}) = f\left(\mathrm{v}_1, \ldots, \mathrm{v}_{j-1}, \mathrm{v}, \mathrm{v}_{j+1}, \ldots, \mathrm{v}_p\right) \qquad for\ all \quad \mathrm{v} \in \mathcal{V}_j$$

is a linear form.

Example

Suppose $\alpha, \rho \in \mathrm{V}^{(2)}$. Define a function $\beta: \mathrm{V}^4 \to \mathbf{F}$ by then $\beta \in \mathrm{V}^{(4)}$ (which means multilinear)

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\rho(v_3, v_4)$$

Multilinear Forms

Definition

Suppose m is a positive integer.

- An m−linear form α on V is called ***alternating*** if $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m$ is a list of vectors in V with $v_j = v_k$ for some two distinct values of j and k in $\{1, \ldots, m\}$.

- $V_{alt}^{(m)} = \{\alpha \in V^{(m)} : \alpha \text{ is an alternating } m\text{−linear form on } V\}$.

Theorem (3)

$V_{alt}^{(m)}$ is a subspace of $V^{(m)}$.

Proof

Review: Characterization of Linearly Dependent sets

Theorem (4)

An indexed set $S = \{v_1, \dots, v_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \dots, v_{j-1}$.

- Does not say that every vector
- Does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

Alternating multilinear forms and linear dependence

Theorem (5)

Suppose m is a positive integer and α is an alternating m-linear form on V. If $v_1, \ldots, v_m$ is a linearly dependent list in V, then

$$\alpha(v_1, \ldots, v_m) = 0$$

Proof

No nonzero alternating m-linear forms for m > dim V

Theorem (6)

Suppose $m(number\ of\ vectors) > \dim V$. Then there is an alternating m-linear form on V.

Proof

Swapping input vectors in an alternating multilinear form

Theorem (7)

Suppose m is a positive integer, α is an alternating m-linear form on V, and $v_1, \dots, v_m$ is a list of vectors in V. Then swapping the vectors in any switch of $\alpha(v_1, \dots, v_m)$ changes the value of α by a factor of -1.

Okey, clearing up the last detail. Suppose we know that A(e_1, e_2, e_3, e_4, e_5) = 7. What should A(e_3, e_5, e_1, e_2, e_4) be?

A(e_3, e_5, e_1, e_2, e_4) = - A(e_3, e_4, e_1, e_2, e_5)
= A(e_3, e_2, e_1, e_4, e_5)
= - A(e_1, e_2, e_3, e_4, e_5) = -7

What if we did the switching in a different order? Would we get the same sign? It turns out that, yes, we would!

Permutation

Definition

Suppose m is a positive integer.

A permutation of $(1, \ldots, m)$ is a list $(j_1, \ldots, j_m)$ that contains each of the numbers $1, \ldots, m$ exactly once.

The set of all permutations of $(1, \ldots, m)$ is denoted by perm m.

Example

- What we need to show is that there is a way to assign a sign to every permutation of $\{1,2,3,\ldots,k\}$ such that, switching the order of any two elements, switches the sign. For example:

$$\begin{array}{lrlr} (1,2,3) \rightsquigarrow & 1 & (1,3,2) \rightsquigarrow & -1 \\ (2,1,3) \rightsquigarrow & -1 & (2,3,1) \rightsquigarrow & 1 \\ (3,1,2) \rightsquigarrow & 1 & (3,2,1) \rightsquigarrow & -1 \end{array}$$

Here is the rule: The sign of $(\sigma(1),\sigma(2),\ldots,\sigma(k))$ is

$$(-1)^{\#\{(i,j)\ :\ i<j \text{ and } \sigma(i)>\sigma(j)\}}.$$

$$A(e_{j_1},e_{j_2},\ldots,e_{j_k}) = \mathtt{sign}(\sigma)A(e_{j_{\sigma(1)}},e_{j_{\sigma(2)}},\ldots,e_{j_{\sigma(k)}}).$$

Permutation

Definition

The $sign$ of a permutation $(j_1, \dots, j_m)$ is defined by

$$sign(j_1, \dots, j_m) = (-1)^N$$

Where N is the number of pairs of integers (k, l) with $1 \le k < l \le m$ such **that k appears after l in the list $(j_1, \dots, j_m)$.**

Example

The permutation $(1, \dots, \mathrm{m})$ [no changes in the natural order] has sign 1.

The only pair of integers (k, l) with $\mathrm{k} < \mathrm{l}$ such that k appears after l in the list $(2,1,3,4)$ is $(1,2)$. Thus the permutation $(2,1,3,4)$ has sign -1.

In the permutation $(2,3, \dots, \mathrm{m}, 1)$, the only pairs (k, l) with $\mathrm{k} < \mathrm{l}$ that appear with changed order are $(1,2), (1,3), \dots, (1, \mathrm{m})$. Because we have $\mathrm{m} - 1$ such pairs, the sign of this permutation equals $(-1)^{\mathrm{m}-1}$.

Permutations and alternating multilinear forms

Theorem (8)

Suppose m is a positive integer and $\alpha \in V_{alt}^{(m)}$. Then

$$\alpha\left(v_{j_1}, \ldots, v_{j_m}\right) = sign(j_1, \ldots, j_m)\alpha(v_1, \ldots, v_m)$$

for every list $v_1, \ldots, v_m$ of vectors in V and all $(j_1, \ldots, j_m) \in perm\, m$.

Proof

Formula for (dim V)-linear alternating forms on V

Theorem (9)

Let $n = \dim V$. Suppose $e_1, \ldots, e_n$ is a basis of V and $v_1, \ldots, v_n \in V$. For each $k \in \{1, \ldots, n\}$, let $b_{1,k}, \ldots, b_{n,k} \in \boldsymbol{F}$ be such that

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j$$

Then

$$\boldsymbol{\alpha(v_1, \ldots, v_n) = \alpha(e_1, \ldots, e_n) \sum_{(j_1, \ldots, j_n) \in perm(n)} \big(sign(j_1, \ldots, j_n)\big) b_{j_1,1} \ldots b_{j_n,n}}$$

for every alternating n-linear form α on V.

Proof

07

Matrix Determinant

Definition

- In **Theorem (9)** if we consider $\alpha(e_1, .., e_n) = 1$ then we can say that determinant is a multilinear alternating form (volume of the transformed vectors.)
- **Determinant is defined for square matrices.**

Definition

Suppose that n is a positive integer and A is an n-by-n square matrix. Then:

$$detA = \sum_{(j_1,\dots,j_n)\in perm(n)} \big(sign(j_1, \dots, j_n)\big) A_{j_1,1} \dots A_{j_n,n}$$

Definition

Note

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |\det A|.\{area\ of\ S\}$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |\det A|.\{volume\ of\ S\}$$

Example

Example

Determinant of 2*2 matrix

Determinant of 3*3 matrix

V($\begin{bmatrix} a & c \\ b & d \end{bmatrix}$) = V($\begin{bmatrix} a \\ b \end{bmatrix}$, $\begin{bmatrix} c \\ d \end{bmatrix}$)

V(a $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, c $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

a V($\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, c $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix}$) +$b$V($\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, c $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix}$)

ac V($\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$) + a$d$ V($\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$) + bc V($\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) + bd V($\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$)+0 + a$d$ V($\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$) + bc V($\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) + 0

=ad – bc V($\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$) = ad – bc

=> V($\begin{bmatrix} a & c \\ b & d \end{bmatrix}$) = ad - bc

Famous Formula of Determinant

Determinant

Example

Let $n = \dim V$.

- If I is the identity operator on V, then $\alpha_1 = \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus, $\det I = 1$.
- More generally, if $\lambda \in \boldsymbol{F}$, then $\alpha_{\lambda I} = \lambda^n \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus, $\det(\lambda I) = \lambda^n$.
- Still more generally, if $T \in \mathcal{L}(V)$ and $\lambda \in \boldsymbol{F}$, then $\alpha_{\lambda T} = \lambda^n \alpha_T = \lambda^n (\det T)\alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus, $\det(\lambda T) = \lambda^n \det T$.

Definition of Submatrix A_{ij}

Definition

For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A

For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

A_{12} is

$$A_{12} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 0 & 7 \\ 0 & -2 & 0 \end{bmatrix}$$

Recursive Definition of Determinant

Definition

The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm\, a_{1j} \det(A_{1j})$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of A. In symbols,

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n})$$
$$= \sum_{j=1}^{n} a_{1j} \underbrace{(-1)^{1+j}\det(A_{1j})}_{C_{1j}}$$

Recursive Definition of Determinant

- 2×2 matrix $\qquad |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}|$$

$$= a \begin{vmatrix} \square & \square \\ \square & d \end{vmatrix} - b \begin{vmatrix} \square & \square \\ c & \square \end{vmatrix}$$

$$= ad - bc$$

Example

$$\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1) \times (1) - (2) \times (-3) = 5$$

Recursive Definition of Determinant

- 3×3 matrix $\qquad |A| = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}|A_{ij}| \qquad i = 1$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \quad |A|$$

$$= (-1)^{1+1}a_{11}|A_{11}| + (-1)^{1+2}a_{12}|A_{12}| + (-1)^{1+3}a_{13}|A_{13}|$$

$$= a\begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b\begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c\begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

Cofactor

Definition

Given $A = [a_{ij}]$, the (i, j)-cofactor of A is the number C_{ij} given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Which is a cofactor expansion across the first row of A.

Cofactor Expansion

Important

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactor is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Cofactor Expansion

- The matrix of cofactors is called the adjugate (or classical adjoint) of A, denoted by **adj** $\boldsymbol{A}$.

- Example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$adj\ A = C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Cofactor Expansion

Example

$$A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} 5 & 4 \\ 3 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = -36$$

$$|A| = -0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -36$$

05

Linking Determinant to other Course Concepts

Gaussian Elimination & Determinant

Note

- Row operations
- Let A be a square matrix.
- If a multiple of one row of A is added to another row to produce a matrix B, then $\det(B) = \det(A)$
- If two rows of A are interchanged to produce B, then $\det(B) = -\det(A)$
- If one row of A is multiplied by k to produce B, then $\det(B) = k.\det(A)$

Matrices Multiplication

Theorem (10)

if A and B are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$

Transformations

Theorem (11)

Show that the determinant, $det: \mathcal{M}n\ (\mathbb{F}) \to \mathbb{F}$ is not a linear transformation when $n \geq 2$

In general, $\det(A+B) \neq \det(A) + \det(B)$

Matrix Inverse

Theorem (12)

A square matrix A is invertible if and only if $\det(A) \neq 0$

Theorem (13)

The determinant of the inverse of an invertible matrix is the inverse of the determinant.

Proof

$$AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1 \Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = |A|^{-1}$$

Matrix Determinant

Example

- ❑ Compute $\det(A)$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

- ❑ The determinant of orthogonal matrix is …

QR Decomposition

Note

- If $A \in \mathcal{M}_n$ has QR decomposition $A = UT$ with $U \in \mathcal{M}_n$ unitary and $T \in \mathcal{M}_n$ upper triangular, then
$$|\det(A)| = t_{1,1}.\, t_{2,2} \dots t_{n,n}.$$

Example

$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ has QR decomposition $A = UT$ with $U = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

08

Cramer’s Rule

Cramer's Rule

- $Ax = b$ and A is invertible
- $A = [A_1 \quad \dots \quad A_n] \qquad I = [e_1 \quad \dots \quad e_n]$

$$AI = A \implies A[e_1 \quad \dots \quad e_n] = [Ae_1 \quad \dots \quad Ae_n] = [A_1 \quad \dots \quad A_n]$$

$$A\overbrace{[e_1 \quad e_2 \quad \dots \quad x \quad \cdots \quad e_n]}^{I_j(x)} = [Ae_1 \quad Ae_2 \quad \dots \quad Ax \quad \cdots \quad Ae_n]$$

$$= \underbrace{[A_1 \quad A_2 \quad \dots \quad b \quad \cdots \quad A_n]}_{A_j(b)}$$

$$|I_2(x)| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = x_2 \implies |I_j(x)| = x_j$$

$$AI_j(x) = A_j(b) \implies |A||I_j(x)| = |A_j(b)| \implies x_j = \frac{|A_j(b)|}{|A|}$$

Cramer's Rule

Note

- Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{|A_i(\mathbf{b})|}{|A|}, \qquad i = 1,2,\dots,n$$

Example

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \Rightarrow \quad x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}} = \frac{-12}{-3} = 4$$

A Formula for A^{-1}

The j-th column of A^{-1} is a vector x that satisfies $\quad Ax = e_j$

By Cramer's rule $\quad \{(\mathrm{i},\mathrm{j}) - \text{entry of } A^{-1}\} = x_{ij} = \frac{|A_i(e_j)|}{|A|}$

$$\left|A_i(e_j)\right| = (-1)^{i+j}\left|A_{ij}\right|$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{2n} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\, adj\ A$$

$$\begin{cases} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{cases} \Rightarrow A^{-1} = \frac{1}{|A|}[C_{ij}]^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

09

Determinant Properties

Good News!

If you understand the properties of determinant as an alternating multilinear form, you can skip the following slides, because you know all of them!! 😁

Properties

- (1) If one row or column is zero, then determinant is zero

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

  - Determinant of zero matrix is…

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

Properties

- (2) If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|A| = -1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

Properties

- (3) If two rows or columns of matrix are interchanged, the sign of determinant is changes!
- (4) $\det(I) = 1$

Properties

- (5) Row and Column Operations
  - If a multiple of one row/column of A is added to another row/column to produce a matrix B, then $\det(A) = \det(B)$.

Proof?

Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

Properties

- (6) If A is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of A.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \qquad \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$

  - Determinant of identity matrix is…

- U is unitary, so that $|\det(U)|=I$

Properties

- (7) If a column or row is multiply to k then determinant is multiply to k.

$$\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- $|kA_{n \times n}| = k^n |A_{n \times n}|$

Properties

- (8) If a row/column is multiple of another row/column then determinant is …..

Properties

- (9) If columns/rows of matrix are linear dependent then its determinant is zero
- (10) If columns/rows of matrix are linear dependent if and only if its determinant is zero.
- (11) Transposing a matrix does not change the determinant.

Reference

- Chapter 3: Linear Algebra and Its Applications, David C. Lay.
- Chapter 9: Part B and C: Linear Algebra Done Right, Sheldon Axler.