Linear Algebra

Diagonalization

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu
Maryam Ramezani maryam.ramezani@sharif.edu

Table of contents

01

Introduction

Conclusion from pervious theorems

Theorem (1)

- Theorem “The eigenvalues of a triangular (upper/lower/diagonal) matrix are the entries on its main diagonal.“ can leads to if we have matrix A and B that $D = B^{-1}AB$ be a diagonal matrix:

$$\det(\lambda I - D) = \det(\lambda I - \mathrm{A})$$

Proof?

Similarity and Diagonalizable

Definition

Two n-by-n matrices A and B are called similar if there exists an invertible n-by-n matrix Q

such that

$$A = Q^{-1}BQ$$

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix D: $\mathrm{D} = Q^{-1}AQ$,
that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D.

02

Similarity

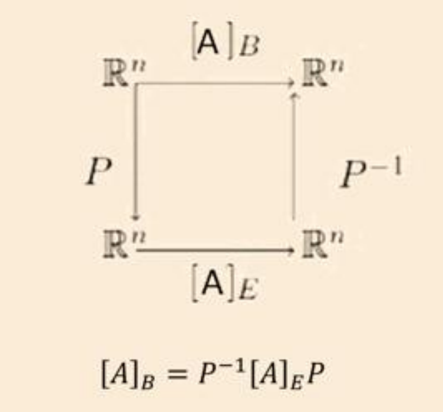
Relation between similar matrix and change of basis!

Note

- A square matrix for a linear transform

$$A: n \times n \qquad T: R^n \to R^n \quad \Rightarrow \quad \boldsymbol{Aa = b} \qquad a, b \in R^n$$

$$\left.\begin{matrix} a = P\bar{a} \\ \mathrm{b} = P\bar{b} \end{matrix}\right\} \Rightarrow \mathrm{A}P\bar{a} = P\bar{b} \Rightarrow \underbrace{P^{-1}AP}_{\bar{A}}\,\bar{a} = \bar{b} \;\Rightarrow\; \bar{A}\bar{a} = \bar{b}$$



- Linear transform in new basis $\bar{A} = P^{-1}AP$
- $\bar{A}$ is the standard matrix of linear transform in new basis.
- Similarity Transformation

Think!

Warnings

1. The matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

are not similar even though they have the same eigenvalues.

2. Similarity is not the same as row equivalence. (If A is row equivalent to B, then $B = EA$ for some invertible matrix E.) Row operations on a matrix usually change its eigenvalues.

- A matrix is a similarity invariant, meaning it remains unchanged under a similarity transformation.

- Why trace is a similarity invariant?
- Why rank is a similarity invariant?

Facts

Theorem (2)

- Similar matrices have:
  - same determinant
  - equal characteristic equations
  - same trace
  - same rank
  - inverse of A and B are similar (if exists)

Proof?

Another Notation

- With similarity transformation Q, matrix A changed to a diagonal matrix $diag(\lambda_{1,},\lambda_2)$
- Matrix A has n linear independent eigenvectors

- $$[Aq_1 \quad Aq_2 \quad \cdots \quad Aq_n] = \underbrace{[q_1 \quad q_2 \quad \cdots \quad q_n]}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}}_{\Lambda}$$
- $A\,[q_1 \; q_2 \; \cdots \; q_n] = Q\Lambda \Rightarrow AQ = Q\Lambda$
- $\Lambda = Q^{-1}AQ^T$
- $A = Q\Lambda Q^{-1}$

Find matrix Q in similarity formula

Note

Two n-by-n matrices A and B are called similar if there exists an invertible n-by-n matrix Q such that $A = Q^{-1}BQ$. One solution for Q is the matrix whose columns are the eigenvectors of B.

Example

Find the similarity matrix of A

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{1} \\ -\boldsymbol{i} & \boldsymbol{i} \end{bmatrix}$$

03

Eigenvalue Multiplicity

Algebraic/ Geometric multiplicity

Example

Find the eigenvalues with their repetition and eigenvectors:

- ❑ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- ❑ $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Definition

- ❑ The number of times an eigenvalue occurs as the root of the matrix's characteristic polynomial is known as its algebraic multiplicity.
- ❑ The geometric multiplicity of the eigenvalue is defined as the dimension of eigenspace associated with that eigenvalue.
- ❑ The geometric multiplicity of an eigenvalue λ is the dimension of its eigenspace, which is the same as the number of linearly independent eigenvectors associated with λ.

Algebraic/ Geometric multiplicity

Theorem (3)

Geometric multiplicity of λ <= Algebraic multiplicity of λ

Proof?

04

Diagonalization

Diagonalizable

Definition

A matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = QDQ^{-1}$ for some invertible matrix Q and some diagonal matrix D. The columns of Q is called an eigenvector basis of $\mathbb{R}^n$.

Example

$\begin{pmatrix} -12 & 15 \\ -10 & 13 \end{pmatrix}$ is diagonalizable because it equals $\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}^{-1}$

$\begin{pmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{pmatrix}$ is diagonalizable because it equals $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}$

$\frac{1}{5}\begin{pmatrix} -8 & -9 \\ 6 & 13 \end{pmatrix}$ is diagonalizable because it equals $\frac{1}{2}\begin{pmatrix} -1 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\left(\frac{1}{2}\begin{pmatrix} -1 & -3 \\ 2 & 1 \end{pmatrix}\right)^{-1}$

$\begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$ is diagonalizable because it equals $\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}^{-1}$.

Diagonalizable

Theorem (4)

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Proof?

Corollary

- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Diagonalizable

Theorem (5)

Let A be an $n \times n$ matrix. The following are equivalent:

- A is diagonalizable.
- The sum of the geometric multiplicities of the eigenvalues of A is equal to n .
- The sum of the algebraic multiplicities of the eigenvalues of A is equal to n , and for each eigenvalue, the geometric multiplicity equals the algebraic multiplicity.

Proof?

Power of matrix

Theorem (6)

Find A^n

Proof?

Diagonalizable and Non-Diagonalizable Matrices

- Distinct eigenvalues -> eigenvectors are Linear Independent
- Duplicate eigenvalues -> 🧐 🧐

- <u>**Not all matrices are diagonalizable**</u>.
  - Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- The diagonalizing matrix S <u>**is not unique**</u>. $AS = SD$

Diagonalizable and Non-Diagonalizable Matrices

- ❑ Distinct eigenvalues -> eigenvectors are Linear Independent
- ❑ Duplicate eigenvalues -> 🧐 🧐

- Not all matrices are diagonalizable.
  - Example:

[0 1]

🧐 🧐

- The dia

Diagonalizable and Non-Diagonalizable Matrices

- For matrix $A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$
  - Its eigenvalues are −2, −2 and −3 (repeated eigenvalues)

$$AS = SD$$

$$\begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -6 \\ 0 & -4 & 3 \\ 0 & 6 & -9 \end{pmatrix}$$

Diagonal Matrix

S is not invertible! => A is non-diagonalizable

Geometric multiplicity of -2= 1

Algebraic multiplicity of -2 =2

Diagonalizable and Non-Diagonalizable Matrices

- For matrix $B = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$
  - Its eigenvalues are −2, −2 and −3 (repeated eigenvalues)

$$BR = RD$$

$$\begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 1 & 2 \\ -3 & 0 & -1 \\ 0 & 3 & 3 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} = \begin{pmatrix} -8 & -2 & -6 \\ 6 & 0 & 3 \\ 0 & -6 & -9 \end{pmatrix}$$

Diagonal Matrix

Geometric multiplicity of -2= 2

Algebraic multiplicity of -2 =2

R is invertible!

R is not unique! $\begin{pmatrix} 5 & 3 & 2 \\ -3 & -3 & -1 \\ 3 & -3 & 3 \end{pmatrix}$

Diagonalizable and Non-Diagonalizable Matrices

- For matrix $B = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \end{pmatrix}$

So, what's going on here?

$\begin{pmatrix} 4 & 8 & - \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$

$\begin{pmatrix} 0 & 3 & 3 \end{pmatrix}$

$\begin{pmatrix} 0 & 3 & 3 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & -3 \end{pmatrix}$

$\begin{pmatrix} 0 & -6 & -9 \end{pmatrix}$

Diagonal Matrix

Geometric multiplicity of -2= 2

Algebraic multiplicity of -2 =2

R is invertible!

R is not unique! $\begin{pmatrix} 5 & 3 & 2 \\ -3 & -3 & -1 \\ 3 & -3 & 3 \end{pmatrix}$

Conclusion
Diagonalizable vs. Non-Diagonalizable Matrices

(1) Distinct Eigenvalues

- Any $n \times n$ matrix with n distinct eigenvalues is always diagonalizable.
  - Reason: each eigenvalue yields exactly one independent eigenvector, giving a full set of n independent eigenvectors.

Conclusion
Diagonalizable vs. Non-Diagonalizable Matrices

(2) Repeated Eigenvalues

Diagonalizability depends on the number of independent eigenvectors.

The matrix is **diagonalizable iff**

$$\text{geometric multiplicity} = \text{algebraic multiplicity} \quad \text{for every } \lambda.$$