Linear Algebra

Matrix Inverse

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Table of contents

01

Introduction to Inverse with Square Matrix

Square Matrix Inverse

Definition

For $A \in M_{n\times n}$, if there exists a matrix $B \in M_{n\times n}$ such that $AB = BA = I_n$, then:

- A is invertible (or nonsingular)
- B is the inverse of A
- The inverse of A is denoted by $B = A^{-1}$
- A square matrix that does not have an inverse is called non-invertible (or singular)

Theorem

The inverse of a square matrix is unique.

Square matrix inverse and column independence

Theorem

A square matrix is invertible if and only if its columns are linearly independent

Proof

- Matrix is invertible => Columns are linearly independent
- Columns are linearly independent => Matrix is invertible

02

Left Inverse

Left Inverse

Definition

A number x that satisfies $xa = 1$ is called the inverse of a

Inverse (i.e., $\frac{1}{a}$) exists if and only if $a \neq 0$, and is unique

A matrix X that satisfies $XA = I$ is called a left inverse of A

If a left inverse exists we say that A is left-invertible

$$A: m \times n \Longrightarrow I: n \times n \Longrightarrow X: n \times m$$

Example

The matrix $A = \begin{bmatrix} -3 & -4 \\ 4 & 6 \\ 1 & 1 \end{bmatrix}$

Has two different left inverses:

$B = \frac{1}{9}\begin{bmatrix} -11 & -10 & 16 \\ 7 & 8 & -11 \end{bmatrix}$, $C = \frac{1}{2}\begin{bmatrix} 0 & -1 & 6 \\ 0 & 1 & -4 \end{bmatrix}$

Solving linear equations with a left inverse

Method

- Suppose $Ax = b$, and A has a left inverse C
- Then $Cb = C(Ax) = (CA)x = Ix = x$
- So multiplying the right–hand side by a left inverse yields the solution

Left inverse of vector

Note

A non-zero column vector always has a left inverse.

Left inverse is not unique.

Example

a= $\begin{bmatrix}1\\0\\3\end{bmatrix}$ Two ways: $(1)a^{-1} = \frac{1}{a_i}e_i^T where\ a_i \neq 0 \quad (2)a^T a = 1 \Rightarrow \frac{a^T}{||a||^2}$

Matrix with orthonormal columns $A^{-1} = A^T$

Lemma

Row vector does not have left inverse

$$A = \begin{bmatrix}1 & 0 & 3\end{bmatrix}$$

Think about rank(BA) , rank(I) with this theory: $rank(BA) \leq \min(rank(A), rank(B))$

Left inverse and column

Theorem

A matrix is left-invertible if and only if its columns are linearly independent

Proof

- Has left invert => Columns are linearly independent
- Columns are linearly independent => Has left invert

Left inverse and column independence

Theorem

If A has a left inverse C then the columns of A are linearly independent
We'll see later that the converse is also true, so:
A matrix is left-invertible if and only if its columns are linearly independent
Matrix generalization of
A number is invertible if and only if it is nonzero
From Previous Theorem
Left-invertible matrices are all tall or square
Wide matrix is not always left invertible
Tall or square matrices can be left invertible

Example

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -2 & -1 \\ 1 & 3 & 4 \\ -2 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

03

Right Inverse

Right inverse and row independence

Theorem

A matrix is right-invertible if and only if its rows are linearly independent

Proof

Right inverses

Definition

A matrix X that satisfies $AX = I$ is a right inverse of A

If a right inverse exists, we say that A is right-invertible

A is right-invertible if and only if A^T is left-invertible:

$$AX = I \Longrightarrow (AX)^T = I \Longrightarrow X^T A^T = I$$

So, we conclude:

A is right invertible if and only if its rows are linearly independent

Right-invertible matrices are wide or square

Solving linear equations with a right inverse

Method

- Suppose A has a right inverse B
- Consider the (square or underdetermined) equations of $Ax = b$
- $x = Bb$ is a solution:
- $Ax = A(Bb) = (AB)b = Ib = b$
- So $Ax = b$ has a solution for any b

Example

Same A, B, C in last example.
C^T and B^T are both right inverses of A^T
Under-determined equations $A^T x = (1, 2)$ has (different) solutions.
$B^T(1,2) = (\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}), \quad C^T(1,2) = \left(0, \frac{1}{2}, -1\right)$
there are many other solutions as well

Conclusion: Left and Right Inverse

Linear equations and matrix inverse

Definition

Left-Invertible matrix: if X is a left inverse of A, then

$$Ax = b \Longrightarrow x = XAx = Xb$$

There is at most one solution using X (if there is a solution, it must be equal to Xb)

We must know in advance that there exists at least one solution

Why "at most"??

$XA = I$

$$\begin{cases} -y_1 + y_2 = -4 \\ 0y_1 - y_2 = 3 \\ 2y_1 + y_2 = 0 \end{cases} \qquad A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 5 & 2 \end{bmatrix} \qquad \left[\begin{array}{cc|c} -1 & 1 & -4 \\ 0 & -1 & 3 \\ 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$$

Linear equations and matrix inverse

Note

- If the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent, and if a matrix $\boldsymbol{B}$ exists such that $\boldsymbol{BA} = \boldsymbol{I}$, then the system of equations has a unique solution, namely $\boldsymbol{x} = \boldsymbol{Bb}$.
- **Right-inversible matrix:** if X is a right inverse of A, then there is **at least one** solution (x=Xb):

$$x = Xb \implies Ax = AXb = b$$

- To pursue these ides further, suppose that again we want to solve a system of linear equations, $\boldsymbol{Ax} = \boldsymbol{b}$. Assume now that we have another matrix, $\boldsymbol{B}$, such that $\boldsymbol{AB} = \boldsymbol{I}$. Then we can write $\boldsymbol{A}(\boldsymbol{Bb}) = (\boldsymbol{AB})\boldsymbol{b} = \boldsymbol{Ib} = \boldsymbol{b}$; hence $\boldsymbol{Bb}$ solves the equations $\boldsymbol{Ax} = \boldsymbol{b}$. This conclusion did not require an a prior assumption that a solution exist; we have produced a solution. The argument does not reveal whether $\boldsymbol{Bb}$ is the only solution. There may be others.
- **Invertible matrix:** if A is invertible, then

$$\boldsymbol{Ax} = \boldsymbol{b} \iff \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

There is a unique solution

Conclusion

- System of linear equations $Ax = b$:
  - A right inverse of A, say $AB = I$. Then Bb is a solution, as is verified by nothing $A(Bb) = (AB)b = Ib = b$.
  - Why don't need to check the consistency for using right inverse?
  - A left inverse of A, say $\mathrm{CA} = I$, then we can only conclude that Cb is the sole candidate for a solution; however, it must be checked by substitution to determine whether, in fact, it is a solution

04

Square Matrix Inverse

Inverse

Definition

For $A \in M_{n\times n}$, if there exists a matrix $B \in M_{n\times n}$ such that $AB = BA = I_n$, then:

- A is invertible (or nonsingular)
- B is the inverse of A
- The inverse of A is denoted by $B = A^{-1}$
- A square matrix that does not have an inverse is called non-invertible (or singular)
- **For a square matrix left and right inverse are the same. Rows and columns are linear independent.**

Square matrix inverse

Theorem

For a square matrix, the right and left inverse are the same

Proof

Invertible Matrices

$$
\begin{tikzcd}
\text{left-invertible} \arrow[r, Rightarrow, "\substack{\text{Holds in general}\\ \text{(a)}}"] & \text{linearly independent columns} \arrow[d, Rightarrow, "\text{(b)}\ \substack{\text{Holds for}\\ \text{square matrices}}"] \\
\text{linearly independent rows} \arrow[u, Rightarrow, "\text{(b')}"', "\substack{\text{Holds for}\\ \text{square matrices}}"] & \text{right-invertible} \arrow[l, Rightarrow, "\text{(a')}"', "\text{Holds in general}"]
\end{tikzcd}
$$

Gauss-Jordan Elimination for finding the Inverse of a matrix

Method

- Let A be a $n \times n$ matrix:
  - Adjoin the identity $n \times n$ matrix I_n to A to form the matrix $[A : I_n]$.
  - Compute the reduced echelon form of $[A : I_n]$.
- If the reduced echelon form is of the type $[I_n : B]$, then B is the inverse of A.
- If the reduced echelon form is not the type $[I_n : B]$, in that the first $n \times n$ submatrix is not I_n then A has no inverse.

$$[A \quad | \quad I] \text{ Gauss—Jordan elimination } [I \quad | \quad A^{-1}]$$

Important

An $n \times n$ matrix is invertible if and only if its reduced echelon form is I_n.

A is row equivalent to I_n

Inverse (Example)

Example

Find inverse of the following matrix using Gauss-Jordan Elimination:

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

$$AX = I \implies \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies \begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By equating corresponding entries we have:

$$\begin{cases} x_{11} + 4x_{21} = 1 \\ -x_{11} - 3x_{21} = 0 \end{cases} \quad (1)$$
$$\begin{cases} x_{12} + 4x_{22} = 0 \\ -x_{12} - 3x_{22} = 1 \end{cases} \quad (2)$$

This two system of linear equations have the same coefficient matrix, which is exactly the matrix A

Inverse (Example)

Rest of The Example

Using Gauss–Jordan Elimination on the matrix A with the same row operations

$$(1) \Rightarrow \left[\begin{array}{cc|c} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3, x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4, x_{22} = 1$$

Thus $X = A^{-1} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$

Solution for $\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}$ Solution for $\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$

$$\underbrace{\left[\begin{array}{cc|cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right]}_{A \qquad I} \xRightarrow{Guass-Jordan\ elimination} \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]}_{I \qquad A^{-1}}$$

Inverse

Definition

Properties (If A is invertible matrix, k is a positive integer and c is a scalar):

A^{-1} is invertible and $(A^{-1})^{-1} = A$

A^k is invertible and $\left(A^k\right)^{-1} = A^{-k} = (A^{-1})^k$

cA is invertible if $c \neq 0$ and $(cA)^{-1} = \frac{1}{c} A^{-1}$

A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Theorem

If A and B are invertible matrices of order n, then AB is invertible and

$(AB)^{-1} = B^{-1}A^{-1}$

$$(A_1A_2A_3 \cdots A_n)^{-1} = A_n^{-1} \cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

Inverse

Theorem

The solution set K of any system Ax=b of m linear in n unknows is (so is a linear map T with standard matrix A), s is a particular solution:

$$K = s + Null(T_A)$$

Theorem (Using above Theorem)

Let $Ax = b$ be a system of n linear equations in n variable.

The system has exactly one solution $\mathrm{C}b$ if and only if A is invertible and $\mathrm{C} = A^{-1}$.

Invertible Matrix

Definition

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible

Note

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\det A = ad - bc$.

2×2 matrix A is invertible if and only if $\det A \neq 0$.

Elementary Matrices

Definition

Each Elementary Matrix is E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example

Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

Solving square systems of linear equations

Method

- Suppose A is invertible
- For any b, $Ax = b$ has the unique solution

$$x = A^{-1}b$$

- Matrix generalization of simple scalar equation $ax = b$ having solution $x = \left(\frac{1}{a}\right)b$ (for $a \neq 0$)
- Simple-looking formula $x = A^{-1}b$ is basis for many applications

Invertible (Nonsingular) matrices

Conclusion

The following are equivalent for a square matrix A:

- ❑ A is invertible
- ❑ Columns of A are linearly independent
- ❑ Rows of A are linearly independent
- ❑ A has a left inverse
- ❑ A has a right inverse

$$row\,rank(A) = col\,rank(A) = n$$

If any of these hold, all others do

Invertible matrices

Examples

$$I^{-1} = I$$

If $\boldsymbol{Q}$ is orthogonal, i.e. , square with $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$, then $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$

2×2 matrix A is invertible if and only if $A_{11}A_{22} \neq A_{12}A_{21}$

$$A^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}$$

You need to know this formula

There are similar but much more complicated formulas for larger matrices (and no, you do not need to know them)

Consider matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ -3 & -4 & -4 \end{bmatrix}$

A is invertible, with inverse:

$$A^{-1} = \frac{1}{30} \begin{bmatrix} 0 & -20 & -10 \\ -6 & 5 & -2 \\ 6 & 10 & 2 \end{bmatrix}$$

Verified by checking $AA^{-1} = I$ (or $A^{-1}A = I$)

We'll soon see how to compute the inverse

Properties

Properties

- $(AB)^{-1} = B^{-1}A^{-1}$
- If A is nonsingular, then A^T is nonsingular

$$(A^T)^{-1} = (A^{-1})^T \text{ (sometimes denoted } A^{-T}\text{)}$$

- Negative matrix powers: $(A^{-1})^k$ is denoted by A^{-k}
- With $A^0 = I$, Identity $A^kA^l = A^{k+l}$ holds for any integers k, l

Triangular matrices

Theorem

Lower Triangular L with non-zero diagonal entries is invertible

Proof??

Theorem

Upper Triangular R with non-zero diagonal entries is invertible

Proof??

Why Matrix of Change of Basis is invertible?

$Because\ the\ column\ and\ rows\ of\ it\ is\ the\ basis\ so\ they\ are\ linear\ independent\ and\ invertible$

Rank and Inverse

Theorem

Given a square matrix M and its inverse M^{-1}, then M and M^{-1} have the same rank.

Rank and Inverse

Theorem

If A is m × n and B is an n × n invertible matrix, then rank(AB) = rank(A).

Solution:

By theorem $\mathbf{rank(AB) \le min(rank(A), rank(\mathit{B})} \Rightarrow$

$rank(AB) \le rank(A), [*] rank(AB) \le rank(B)$

Also, we can write:

$AB\,B^{-1} = A \Rightarrow rank(A) \le rank(AB)$ [**]

so using [*],[**] then rank(A)=rank(AB)

The Invertible Matrix Theorem

Let A be a square n × n matrix. Then the following are equivalent.

- 1. A is an invertible matrix.
- 2. A is row equivalent to the n × n identity matrix.
- 3. A has n pivot positions.
- 4. The equation Ax = 0 has only the trivial solution.
- 5. The columns of A form a linearly independent set.
- 6. The linear transformation x → Ax is one-to-one.
- 7. The equation Ax = b has at least one solution for each $\mathbf{b} \in \mathbb{R}^n$.
- 8. The columns of A span $\mathbb{R}^n$.
- 9. The linear transformation x → Ax maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
- 10. There is an n × n matrix C such that CA = I.
- 11. There is an n × n matrix D such that AD = I.
- 12. A^T is an invertible matrix.

04

Inverse with QR

QR Decomposition

- A QR decomposition can be created for any matrix — it need not be square and it need not have full rank.
- Every matrix has a QR-decomposition, though R may not always be invertible.

Inverse via QR factorization (square matrix)

Suppose A is square and invertible :

- So its columns are linearly independent
- So Gram-Schmidt gives QR factorization
  - $A = QR$
  - Q is orthogonal $Q^T Q = I$
  - R is upper triangular with positive diagonal entries, hence invertible
- So we have

$$A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T$$

Inverse via QR factorization (square matrix)

Algorithm: Computing Matrix Inverse

Input: $A_{n\times n}$ invertible

Output: $A^{-1}_{n\times n}$

Find QR factorization $A = QR$

$\begin{bmatrix} \bar{q}_1 & \cdots & \bar{q}_n \end{bmatrix} = Q^T$

for $i = 1, \dots, n$ **do**

Solve $Rx_i = \bar{q}_i$ using back substituition

end

$A^{-1} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$

48