Linear Algebra

# Determinant

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee rabiee@sharif.edu
Maryam Ramezani maryam.ramezani@sharif.edu

# Table of contents

# 01

# Introduction

# Determinant of a matrix

The determinant of a $2 \times 2$ matrix $A = [a_{ij}]$ is the number:
Why???

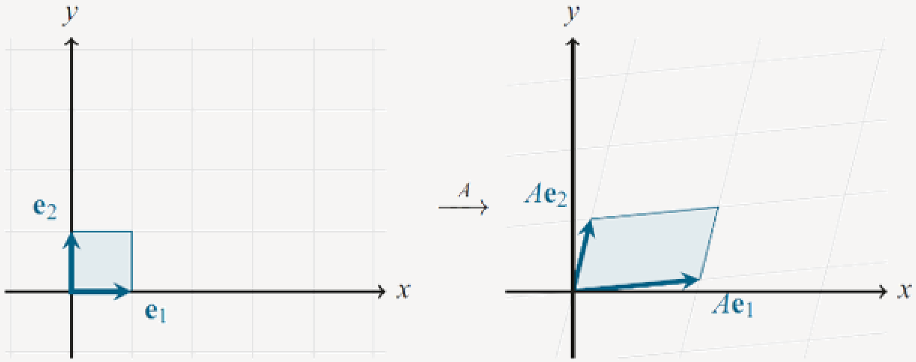
$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

- ❑ The absolute value of the determinant of a matrix measures how much it expands space when acting as a linear transformation. That is, it is the area (or volume, or hypervolume, depending on the dimension) of the output of the unit square, cube, or hypercube after it is acted upon by the matrix.

# Geometric interpretation

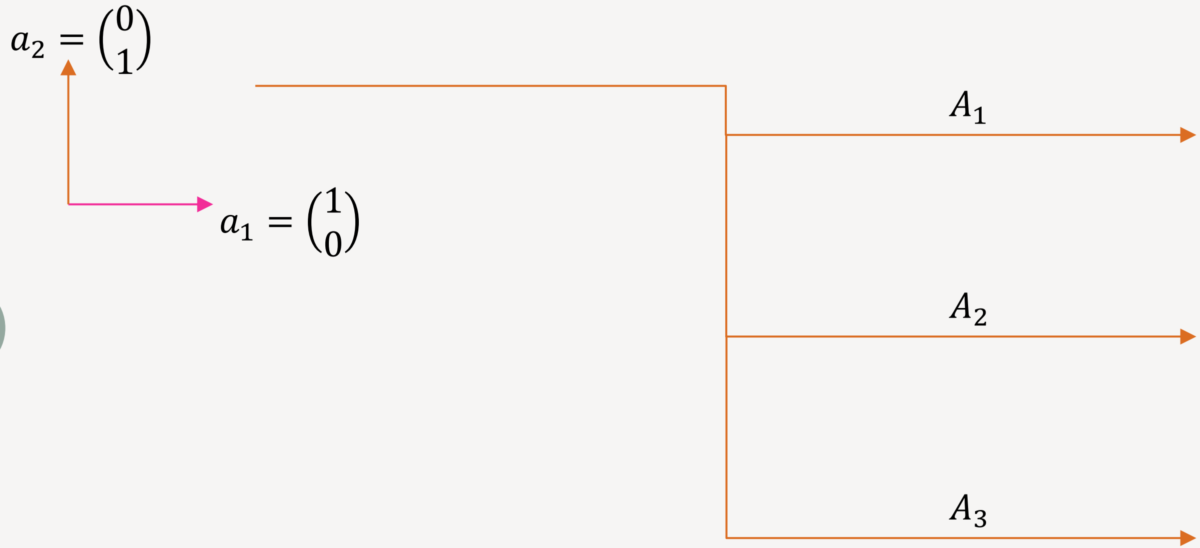
❑ The volume is a n-alternating multilinear map on all n-parallelepipeds such that the volume of standard unit parallelepiped is one.

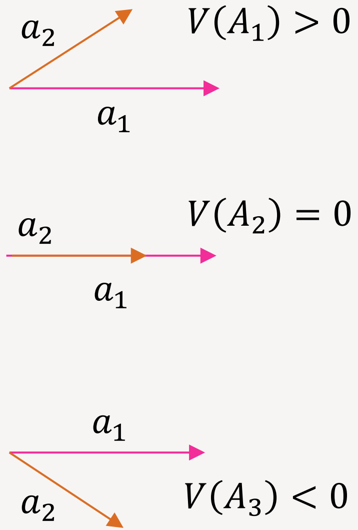
$$\frac{\textit{volume of output region}}{\textit{volume of input region}}$$




A $2 \times 2$ matrix $A$ stretches the unit square (with sides $e_1$ and $e_2$) into a parallelogram with sides $Ae_1$ and $Ae_2$ (the columns of $A$). The determinant of $A$ is the area of this parallelogram.

# Geometric interpretation



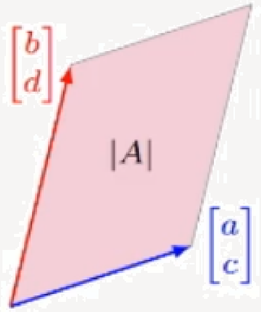



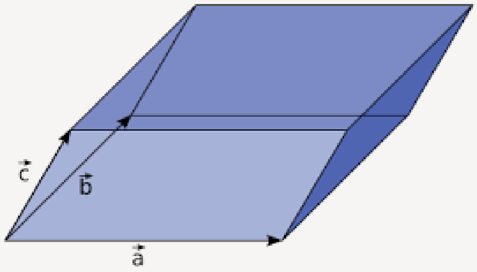

$$A = [a_1 \quad a_2]\,, a_2 = \alpha a_1$$

$$V(a_1, a_2) = -V(a_2, a_1)$$

# Determinants as Area or Volume

- If $A$ is a $2 \times 2$ matrix, the area of the parallelogram determined by the columns of A is det(A)
- If $A$ is a $3 \times 3$ matrix, the volume of the parallelepiped determined by the columns of A is det(A)
- Examples: Volume of $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ It is a rotation with $\boldsymbol{\theta}$ degree



# Volume

## Definition

Every $\boldsymbol{n}$-dimensional parallelepiped with $\{\boldsymbol{a_1}, \dots, \boldsymbol{a_n}\}$ as legs is associated with a real number, called its volume which has the following properties:

If we stretch a parallelepiped by multiplying one of its legs by a scalar $\lambda$, its volume gets multiplied by $\lambda$.

If we add a vector $\omega$ to $i$-th legs of a $n$-dimensional parallelepiped with $\{a_1, \dots, a_i, a_{i+1}, \dots, a_n\}$, then its volume is the sum of the volume from $\{a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n\}$ and the volume of $\{a_1, \dots, a_{i-1}, \omega, a_{i+1}, \dots, a_n\}$.

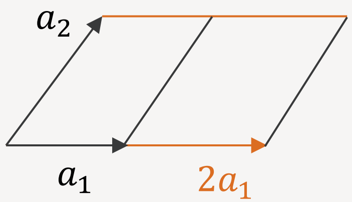
The volume changes sign when two legs are exchanged.

The volume of the parallelepiped with $\{e_1, \dots, e_n\}$ is one.

$$\phi : \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

# Volume

- Example - 2D Case
  - V($a_1$, $a_2$)
  - V($2\overrightarrow{a_1}$, $\overrightarrow{a_2}$) = 2V($a_1$, $a_2$)
  - V(-$a_1$, $a_2$) = - V($a_1$, $a_2$)
  - V(β$a_1$, $a_2$) = - β V($a_1$, $a_2$)



02

# Bilinear Form: Review and Continue

# Bilinear Form over a complex vector

## Definition

Suppose $V$ and $W$ are vector spaces over the same field $\mathbb{C}$. Then a function $\alpha: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

It is linear in its first argument:

$\alpha(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = \alpha(\mathbf{v_1}, \mathbf{w}) + \alpha(\mathbf{v_2}, \mathbf{w})$ and

$\alpha\,(\lambda \mathbf{v_1}, \mathbf{w}) = \lambda\alpha(\mathbf{v_1}, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.

It is conjugate linear in its second argument:

$\alpha\,(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = \alpha(\mathbf{v}, \mathbf{w_1}) + \alpha(\mathbf{v}, \mathbf{w_2})$ and

$\alpha\,(\mathbf{v}, \lambda\mathbf{w_1}) = \bar{\lambda}\alpha(\mathbf{v}, \mathbf{w_1})$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

The set of bilinear forms on $\mathbf{v}$ is denoted by $\mathbf{v}^2$ .

# Alternating bilinear form

## Definition

A bilinear form $\alpha \in V^{(2)}$ is called alternating if

$$\alpha(v, v) = 0$$

for all $v \in V$. The set of alternating bilinear forms on $V$ is denoted by $V_{alt}^{(2)}$.

## Example

Suppose $\varphi, \tau \in V'$. Then the bilinear form $\alpha$ on $V$ defined by is alternating.

$$\alpha(u, \omega) = \varphi(u)\tau(\omega) - \varphi(\omega)\tau(u)$$

# Alternating bilinear form

## Theorem

A bilinear form $\alpha$ on $V$ is alternating if and only if

$$\alpha(u,\omega) = -\,\alpha(\omega,u)$$

For all $u,\omega \in V$.

Proof

# Alternating bilinear form

**Theorem**

The sets $V_{sym}^{(2)}$ and $V_{alt}^{(2)}$ are subspaces of $V^{(2)}$. Furthermore,

$$V^{(2)} = V_{sym}^{(2)} \oplus V_{alt}^{(2)}$$

Proof

03

# Multilinear Form

# Multilinear Forms

## Definition

Suppose $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_p$ are vector spaces over the same field $\mathbb{F}$. A function

$$f : \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_p \to \mathbb{F}$$

is called a multilinear form if, for each $1 \leq j \leq p$ and each $\mathrm{v}_1 \in \mathcal{V}_1, \mathrm{v}_2 \in \mathcal{V}_2, \dots, \mathrm{v}_p \in \mathcal{V}_p$, it is the case that the function $g : \mathcal{V}_j \to \mathbb{F}$ defined by

$$g(\mathrm{v}) = f\left(\mathrm{v}_1, \dots, \mathrm{v}_{j-1}, \mathrm{v}, \mathrm{v}_{j+1}, \dots, \mathrm{v}_p\right) \qquad for\ all \quad \mathrm{v} \in \mathcal{V}_j$$

is a linear form.

## Example

Suppose $\alpha, \rho \in V^{(2)}$. Define a function $\beta: V^4 \to \boldsymbol{F}$ by then $\beta \in V^4$

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\rho(v_3, v_4)$$

16

# Multilinear Forms

## Definition

Suppose $m$ is a positive integer.

- An $m-$linear form $\alpha$ on $V$ is called *alternating* if $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m$ is a list of vectors in $V$ with $v_j = v_k$ for some two distinct values of $j$ and $k$ in $\{1, \ldots, m\}$.

- $V_{alt}^{(m)} = \{\alpha \in V^{(m)}: \alpha \text{ is an alternating } m\text{−linear form on } V\}$.

## Theorem

$V_{alt}^{(m)}$ is a subspace of $V^{(m)}$.

Proof

# Review: Characterization of Linearly Dependent sets

**Theorem**

An indexed set $S = \{v_1, \ldots, v_n\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in $S$ is a linear combination of the others. In fact, if $S$ is linearly dependent and $v_1 \neq 0$, then some $v_j$ (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \ldots, v_{j-1}$.

- Does not say that every vector
- Does not say that every vector in a linearly dependent set is a linear combination of the preceding vectors. A vector in a linearly dependent set may fail to be a linear combination of the other vectors.

# Alternating multilinear forms and linear dependence

**Theorem**

Suppose $m$ is a positive integer and $\alpha$ is an alternating $m$-linear form on $V$. If $v_1, \dots, v_m$ is a linearly dependent list in $V$, then

$$\alpha(v_1, \dots, v_m) = 0$$

Proof

# No nonzero alternating $m$-linear forms for $m > \dim V$

**Theorem**

Suppose $m(number\ of\ vectors) > \dim V$.

Then $\mathbf{0}$ is the only alternating $m$-linear form on $V$.

Proof

# Swapping input vectors in an alternating multilinear form

**Theorem**

Suppose $m$ is a positive integer, $\alpha$ is an alternating $m$-linear form on $V$, and $v_1, \ldots, v_m$ is a list of vectors in $V$. Then swapping the vectors in any switch of $\alpha(v_1, \ldots, v_m)$ changes the value of $\alpha$ by a factor of $-1$.

Okey, clearing up the last detail. Suppose we know that A($e_1$, $e_2$, $e_3$, $e_4, e_5$) = 7. What should A($e_3, e_5$, $e_1$, $e_2$, $e_4$) be?

A($e_3, e_5$, $e_1$, $e_2$, $e_4$) = - A($e_3, e_4$, $e_1$, $e_2$, $e_5$)
= A($e_3, e_2$, $e_1$, $e_4$, $e_5$)
= - A($e_1$, $e_2$, $e_3$, $e_4, e_5$) = -7

What if we did the switching in a different order? Would we get the same sign? It turns out that, yes, we would!

# Permutation

## Definition

Suppose $m$ is a positive integer.

A permutation of $(1, \ldots, m)$ is a list $(j_1, \ldots, j_m)$ that contains each of the numbers $1, \ldots, m$ exactly once.

The set of all permutations of $(1, \ldots, m)$ is denoted by perm $m$.

# Example

- What we need to show is that there is a way to assign a sign to every permutation of $\{1, 2, 3, \ldots, k\}$ such that, switching the order of any two elements, switches the sign. For example:

$$\begin{array}{lcl} (1,2,3) \rightsquigarrow \ \ 1 & \quad & (1,3,2) \rightsquigarrow -1 \\ (2,1,3) \rightsquigarrow -1 & & (2,3,1) \rightsquigarrow \ \ 1 \\ (3,1,2) \rightsquigarrow \ \ 1 & & (3,2,1) \rightsquigarrow -1 \end{array}$$

Here is the rule: The sign of $(\sigma(1), \sigma(2), \ldots, \sigma(k))$ is

$$(-1)^{\#\{(i,j)\ :\ i<j \text{ and } \sigma(i)>\sigma(j)\}}.$$

$$A(e_{j_1}, e_{j_2}, \ldots, e_{j_k}) = \mathtt{sign}(\sigma) A(e_{j_{\sigma(1)}}, e_{j_{\sigma(2)}}, \ldots, e_{j_{\sigma(k)}}).$$

# Permutation

## Definition

The $sign$ of a permutation $(j_1, \dots, j_m)$ is defined by

$$sign(j_1, \dots, j_m) = (-1)^N$$

Where $N$ is the number of pairs of integers $(k, l)$ with $1 \leq k < l \leq m$ such that $k$ appears after $l$ in the list $(j_1, \dots, j_m)$.

## Example

The permutation $(1, \dots, \mathrm{m})$ [no changes in the natural order] has sign 1.

The only pair of integers $(\mathrm{k}, \mathrm{l})$ with $\mathrm{k} < \mathrm{l}$ such that k appears after l in the list $(2,1,3,4)$ is $(1,2)$.

Thus the permutation $(2,1,3,4)$ has sign $-1$.

In the permutation $(2,3, \dots, \mathrm{m}, 1)$, the only pairs $(\mathrm{k}, \mathrm{l})$ with $\mathrm{k} < \mathrm{l}$ that appear with changed order are $(1,2), (1,3), \dots, (1, \mathrm{m})$. Because we have $\mathrm{m} - 1$ such pairs, the sign of this permutation equals $(-1)^{\mathrm{m}-1}$.

# Permutations and alternating multilinear forms

## Theorem

Suppose $m$ is a positive integer and $\alpha \in V_{alt}^{(m)}$. Then

$$\alpha(v_{j_1}, \dots, v_{j_m}) = sign(j_1, \dots, j_m)\alpha(v_1, \dots, v_m)$$

for every list $v_1, \dots, v_m$ of vectors in $V$ and all $(j_1, \dots, j_m) \in perm\ m$.

Proof

# Formula for (dim $V$)-linear alternating forms on $V$

## Theorem

Let $n = \dim V$. Suppose $e_1, \ldots, e_n$ is a basis of $V$ and $v_1, \ldots, v_n \in V$. For each $k \in \{1, \ldots, n\}$, let $b_{1,k}, \ldots, b_{n,k} \in \boldsymbol{F}$ be such that

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j$$

$$v_1 = \begin{bmatrix} a \\ b \end{bmatrix}, v_2 = \begin{bmatrix} c \\ d \end{bmatrix}$$

Then

$$\boldsymbol{\alpha(v_1, \ldots, v_n) = \alpha(e_1, \ldots, e_n) \sum_{(j_1, \ldots, j_n) \in perm(n)} (sign(j_1, \ldots, j_n)) b_{j_1,1} \ldots b_{j_n,n}}$$

for every alternating $n$-linear form $\alpha$ on $V$.

Proof

# Nonzero alternating $n$-linear form $\alpha$ on

## Theorem

The vector space $\alpha_{alt}^{(\dim V)}$ with inputs from vector space $V$ from has dimension one.

Proof

## Theorem

$$\boldsymbol{\alpha(v_1, \dots, v_n) = \sum_{(j_1,\dots,j_n)\in perm(n)} \big(sign(j_1, \dots, j_n)\big)\varphi_{j_1}(v_1) \dots \varphi_{j_n}(v_n)}$$

The verification that $\alpha$ is an $n$-linear form on $V$ is straightforward.

$$\alpha(e_1, .., e_n) = 1$$

04

# Matrix Determinant

# Determinant

## Definition

**Non-square matrices do not have determinants.**

Suppose that $m$ is a positive integer and $T \in \mathcal{L}(V)$. For $\alpha \in V_{alt}^{(m)}$, define $\alpha \in V_{alt}^{(m)}$ by

$$\alpha_T(v_1, \ldots, v_m) = \alpha(Tv_1, \ldots, Tv_m)$$

for each list $v_1, \ldots, v_m$ of vectors in $V$.

$$\alpha_T = (\det T)\alpha$$

The vector space $V_{alt}^{(\dim V)}$ has dimension one.

# Example

$V(\begin{bmatrix} a & c \\ b & d \end{bmatrix}) = V(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix})$

$V(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix})$

$a\,V(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix}) + bV(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, c\begin{bmatrix} 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \end{bmatrix})$

$ac\,V(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) + ad\,V(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) + bc\,V(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) + bd\,V(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix})$

$0 + ad\,V(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) + bc\,V(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) + 0$

$ad - bc\,V(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = ad - bc$

$V(\begin{bmatrix} a & c \\ b & d \end{bmatrix}) = ad - bc$

determinant

# Determinant

## Example

Let $n = \dim V$.

- If $I$ is the identity operator on $V$, then $\alpha_1 = \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det I = 1$.

- More generally, if $\lambda \in \boldsymbol{F}$, then $\alpha_{\lambda I} = \lambda^n \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det(\lambda I) = \lambda^n$.

- Still more generally, if $T \in \mathcal{L}(V)$ and $\lambda \in \boldsymbol{F}$, then $\alpha_{\lambda T} = \lambda^n \alpha_T = \lambda^n (\det T)\alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det(\lambda T) = \lambda^n \det T$.

# Determinant is an alternating

## Theorem

Suppose that $n$ is a positive integer. The map that takes a list $v_1, \dots, v_n$ of vectors in $\boldsymbol{F}^n$ to $\det(v_1, \dots, v_n)$ is an alternating $n$-linear form of $\boldsymbol{F}^n$.

# Matrix Determinant

## Theorem

Suppose that $n$ is a positive integer and $A$ is an $n$-by-$n$ square matrix. Then

$$\boldsymbol{detA = \sum_{(j_1,\dots,j_n)\in perm(n)} \big(sign(j_1,\dots,j_n)\big) A_{j_1,1} \dots A_{j_n,n}}$$

Proof

## Example

Determinant of 2*2 matrix
Determinant of 3*3 matrix

# Definition of Submatrix $A_{ij}$

## Definition

For any square matrix $A$, let $A_{ij}$ denote the submatrix formed by deleting the $i$th row and $j$th column of $A$

For instance, if

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

$A_{12}$ is

$$A_{12} = \begin{bmatrix} 2 & 4 & -1 \\ 3 & 0 & 7 \\ 0 & -2 & 0 \end{bmatrix}$$

# Recursive Definition of Determinant

## Definition

The determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of $n$ terms of the form $\pm a_{1j} \det(A_{1j})$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \dots, a_{1n}$ are from the first row of $A$. In symbols,

$$\begin{aligned}\det(A) &= a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n}) \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})\end{aligned}$$

# Recursive Definition of Determinant

- 2 × 2 matrix $\quad |A| = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}|A_{ij}| \qquad i = 1$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow |A| = (-1)^{1+1}a_{11}|A_{11}| + (-1)^{1+2}a_{12}|A_{12}|$$

$$= a\begin{vmatrix} \square & \square \\ \square & d \end{vmatrix} - b\begin{vmatrix} \square & \square \\ c & \square \end{vmatrix}$$

$$= ad - bc$$

## Example

$$\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1) \times (1) - (2) \times (-3) = 5$$

# Recursive Definition of Determinant

- $3 \times 3$ matrix $\qquad |A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| \qquad i = 1$

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightarrow \quad |A|$$

$$= (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}|$$

$$= a \begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix} - b \begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} + c \begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei + bfg + cdh - afh - bdi - ceg$$

# Recursive Definition of Determinant

**Example**

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{vmatrix} = -5 + 0 + 6 - (25 + 12 + 0) = -36$$

# Cofactor

## Definition

Given $A = [a_{ij}]$ , the $(i, j)$-cofactor of $A$ is the number $C_{ij}$ given by

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

Then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Which is a cofactor expansion across the first row of $A$.

# Cofactor Expansion

**Important**

The determinant of an $n \times n$ matrix $A$ can be computed by a cofactor expansion across any row or down any column. The expansion across the $i$th row using the cofactor is

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the $j$th column is

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

# Cofactor Expansion

## Example

$$A = \begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & 4 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} 5 & 4 \\ 3 & -1 \end{vmatrix} - 0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 5 \\ 5 & 3 \end{vmatrix} = -36$$

$$|A| = -0 \times \begin{vmatrix} 2 & 4 \\ 5 & -1 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & 1 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} = -36$$

# Cramer's Rule

- $Ax = b$ and $A$ is invertible $\qquad A = [A_1 \quad \dots \quad A_n] \qquad I = [e_1 \quad \dots \quad e_n]$

$$AI = A \implies A[e_1 \quad \dots \quad e_n] = [Ae_1 \quad \dots \quad Ae_n] = [A_1 \quad \dots \quad A_n]$$

$$A\overbrace{[e_1 \quad e_2 \quad \dots \quad x \quad \cdots \quad e_n]}^{I_j(x)} = [Ae_1 \quad Ae_2 \quad \dots \quad Ax \quad \cdots \quad Ae_n]$$

$$= \underbrace{[A_1 \quad A_2 \quad \dots \quad b \quad \cdots \quad A_n]}_{A_j(b)}$$

$$|I_2(x)| = \begin{vmatrix} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{vmatrix} = x_2 \implies |I_j(x)| = x_j$$

$$AI_j(x) = A_j(b) \implies |A||I_j(x)| = |A_j(b)| \implies x_j = \frac{|A_j(b)|}{|A|}$$

# Cramer's Rule

## Note

- Let $A$ be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by
- $x_i = \dfrac{|A_i(\mathbf{b})|}{|A|}, \ i = 1,2,\dots,n$

## Example

$$\begin{cases} x_1 - x_2 + 2x_3 = 1 \\ x_1 + x_2 - x_3 = 2 \\ 2x_1 - 3x_2 + x_3 = -1 \end{cases} \implies x_2 = \frac{\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 2 & -3 & 1 \end{vmatrix}} = \frac{-12}{-3} = 4$$

# A Formula for $A^{-1}$

The $j$-th column of $A^{-1}$ is a vector $x$ that satisfies $\quad Ax = e_j$

By Cramer's rule $\quad \{(\mathrm{i},\mathrm{j}) - \text{entry of } A^{-1}\} = x_i = \dfrac{|A_i(e_j)|}{|A|}$

$$|A_i(e_j)| = (-1)^{i+j}|A_{ji}|$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

The matrix of cofactors is called the adjugate (or classical adjoint) of $A$, denoted by adj $A$.

$$\begin{cases} A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ [C_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \end{cases} \Rightarrow A^{-1} = \frac{1}{|A|}[C_{ij}]^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

# A Formula for $A^{-1}$

**Important**

Let $A$ be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{|A|}\, adj\ A$$

05

# Determinant Properties

# Properties

- (1) If one row or column is zero, then determinant is zero

$$\begin{vmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = 0$$

  - Determinant of zero matrix is…

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

$$\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

# Properties

- $(2)$ If two rows or columns of matrix are same, then determinant is zero.

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & -2 & 3 \\ 5 & 3 & -1 \end{bmatrix}$$

$$|A| = +1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} - (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|A| = -1 \times \begin{vmatrix} -2 & 3 \\ 3 & -1 \end{vmatrix} + (-2) \times \begin{vmatrix} 1 & 3 \\ 5 & -1 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

# Properties

- (3) If two rows or columns of matrix are interchanged, the sign of determinant is changes!
- (4) $\det(I) = 1$

# Properties

- (5) Row and Column Operations
  - If a multiple of one row/column of A is added to another row/column to produce a matrix B, then $\det(A) = \det(B)$.

Proof?

## Example

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & -2 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 6 \\ 1 & 1 & 3 \\ 1 & -1 & 4 \end{vmatrix}$$

# Properties

- $(6)$ If $A$ is a triangular matrix, then $\det(A)$ is the product of the entries on the main diagonal of $A$.

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc \qquad \begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = abc$$

  - Determinant of identity matrix is…

- $U$ is unitary, so that $|\det(U)|=I$

# Properties

- (7) If a column or row is multiply to k then determinant is multiply to k.

$$\begin{vmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

- $|kA_{n\times n}| = k^n |A_{n\times n}|$

# Properties

- (8) If a row/column is multiple of another row/column then determinant is .....

# Properties

- (9) If columns/rows of matrix are linear dependent then its determinant is zero

- (10) If columns/rows of matrix are linear dependent if and only if its determinant is zero.

# Theorem

## Theorem

A square matrix $A$ is invertible if and only if $\det(A) \neq 0$

## Example

Compute $\det(A)$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$

# Echelon form

**Note**

- Row operations
- Let $A$ be a square matrix.
- If a multiple of one row of $A$ is added to another row to produce a matrix $B$, then $\det(B) = \det(A)$
- If two rows of $A$ are interchanged to produce $B$, then $\det(B) = -\det(A)$
- If one row of $A$ is multiplied by $k$ to produce $B$, then $\det(B) = k.\det(A)$

# Echelon form

**Example**

Compute $\det(A)$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

# Determinant of Transpose

**Theorem**

if $A$ is an $n \times n$ matrix, then $\det(A^T) = \det(A)$

# Multiplicative Property

**Theorem**

if $A$ and $B$ are $n \times n$ matrices, then $\det(AB) = \det(A)\det(B)$

Look at pages 27, 34

**Important**

In general, $\det(A + B) \neq \det(A) + \det(B)$

- ❑ The determinant of the inverse of an invertible matrix is the inverse of the determinant

$$AA^{-1} = I \Rightarrow |AA^{-1}| = |I| = 1 \Rightarrow |A||A^{-1}| = 1 \Rightarrow |A^{-1}| = |A|^{-1}$$

- ❑ The determinant of orthogonal matrix is …

# Transformations

## Example

Show that the determinant, $det\colon \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$ is not a linear transformation when $n \geq 2$

# Transformations

## Note

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation determined by a $2 \times 2$ matrix $A$. If $S$ is a parallelogram in $\mathbb{R}^2$, then

$$\{area\ of\ T(S)\} = |\det A| . \{area\ of\ S\}$$

If $T$ is determined by a $3 \times 3$ matrix $A$, and if $S$ is a parallelepiped in $\mathbb{R}^3$, then

$$\{volume\ of\ T(S)\} = |\det A| . \{volume\ of\ S\}$$

# Reference

- Chapter 3: Linear Algebra and Its Applications, David C. Lay.
- Chapter 9: Part B and C: Linear Algebra Done Right, Sheldon Axler.