Linear Algebra

Orthogonality

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01

Orthogonality

Orthogonal Sets

Definition

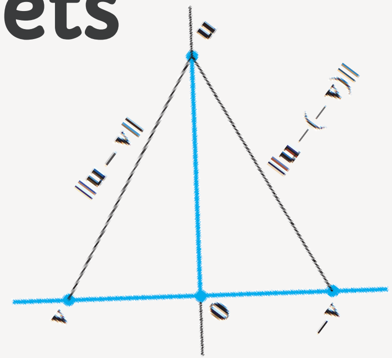
- A set of vectors $\{a_1, \ldots, a_k\}$ in R^n is orthogonal set if each pair of distinct vectors is orthogonal (mutually orthogonal vectors).

A basis B of an inner product space V is called an **orthonormal basis** of V if
a) $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{v} \neq \boldsymbol{w} \in B$, and (mutual orthogonality)
b) $\|\mathbf{v}\| = 1$ for all $\mathbf{v} \in B$. (normalization)

- set of n-vectors $a_1, \ldots, a_k$ are *(mutually) orthogonal* if $a_i \perp a_j$ for $i \neq j$
- They are *normalized* if $\|a_i\| = 1$ for $i = 1, \ldots, k$
- They are *orthonormal* if both hold
- Can be expressed using inner products as $a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Orthogonal Sets

- Geometry



https://youtu.be/dqdSzqsm7bY

- Algebra

Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if $\mathbf{u} \cdot \mathbf{v} = \mathbf{0}$.

Suppose V is an inner product space.
Two vectors $\mathbf{v}, \mathbf{w} \in V$ are called **orthogonal** if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

The Pythagorean Theorem

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$

Orthogonal Sets

Example

- Zero vector is orthogonal to every vector in vector space V
- The standard basis of $\mathbb{R}^n$ or $\mathbb{C}^n$ is an orthogonal set with respect to the standard inner product.

Orthogonal Sets

Theorem

If $S = \{a_1, \ldots, a_k\}$ is an orthogonal set of nonzero vectors in R^n, then S is linearly independent and is a basis for the subspace spanned by S.

Proof

If k = n, then prove that S is a basis for R^n

Linear combinations of orthonormal vectors

Corollary

- A simple way to check if an n−vector y is a linear combination of the orthonormal vectors $a_1, \dots, a_k$, if and only if:

$$y = (a_1^T y)a_1 + \ \dots + (a_k^T y)a_k$$

- For orthogonal vectors $a_1, \dots, a_k$:

$$y = c_1 a_1 + \cdots + c_k a_k$$

$$c_j = \frac{y.a_j}{a_j.a_j}$$

Orthonormal vectors

Theorem

If $S = \{a_1, \ldots, a_k\}$ is an orthogonal set of nonzero vectors in R^n, then S is linearly independent and is a basis for the subspace spanned by S.

Proof

If k = n, then prove that S is a basis for R^n

Orthonormal vectors

Theorem

Independence–dimension inequality

If the n–vectors $a_1, \dots, a_k$ are linearly independent, then $k \leq n$.

- Orthonormal sets of vectors are linearly independent
- By independence-dimension inequality, must have $k \leq n$
- When $k = n, a_1, \dots, a_n$ are an *orthonormal basis*

Example

- Standard unit n-vectors $e_1, .., e_n$
- The 3-vectors

$$\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- The 2-vectors shown below

- The standard basis in $P_n(x)$ $[-1,1]$ (be the set of real-valued polynomials of degree at most n.)

Linear combinations of orthonormal vectors

Example

Write x as a linear combination of a_1, a_2, a_3?

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, a_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, a_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

02

Orthogonal Subspaces

Definition

- Two subspaces W_1 and W_2 of the same space V are orthogonal, denoted by $W_1 \perp W_2$, if and only if each vector $w_1 \in W_1$ is orthogonal to each vector $w_2 \in W_2$ for all w_1, w_2 in W_1, W_2 respectively:

$$< w_1, w_2 > = 0$$

Example

If the bases of two subspaces are orthogonal, it implies that the subspaces themselves are orthogonal.

03

Orthogonal Complements

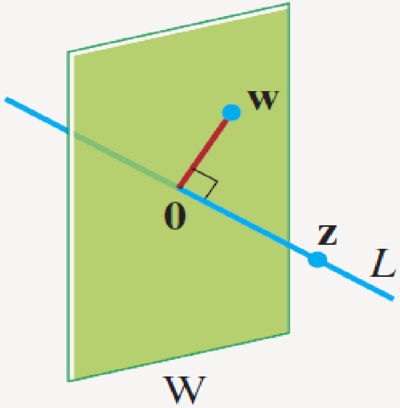
Definition

- If a vector z is orthogonal to every vector in a subspace W of R^n, then z is said to be orthogonal to W.
- **The set of all vectors z that are orthogonal** to W is called the orthogonal complement of W and is denoted by $W^{\perp}$

Example

W be a plane through the origin in $\mathbb{R}^3$.

$L = W^{\perp}$and $W = L^{\perp}$



Orthogonal Complements

Theorem

$W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Theorem

$W^{\perp} \cap W = \{\mathbf{0}\}$.

Important

We emphasize that W_1 and W_2 can be orthogonal without being complements.

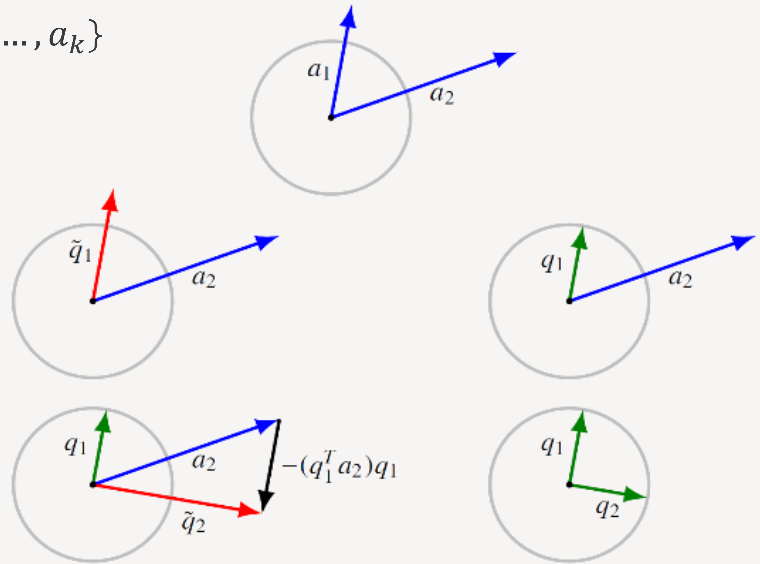
$W_1 = span((1,0,0))$ and $W_2 = span((0,1,0))$.

04

Gram–Schmidt Algorithm

Gram–Schmidt (orthogonalization) algorithm

- Find orthonormal basis for span $\{a_1, a_2, \dots, a_k\}$
- Geometry:



Gram–Schmidt (orthogonalization) algorithm

- Find orthonormal basis for span $\{a_1, a_2, \dots, a_k\}$
- Algebra:

1) $q1 = \frac{a_1}{\|a_1\|}$

2) $\widetilde{q_2} = a_2 - (q_1^T a_2)q_1 \rightarrow q_2 = \frac{\widetilde{q_2}}{\|\widetilde{q_2}\|}$

3) $\widetilde{q_3} = a_3 - (q_1^T a_3)q_1 - (q_2^T a_3)q_2 \rightarrow q_3 = \frac{\widetilde{q_3}}{\|\widetilde{q_3}\|}$

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k) $\widetilde{q_k} = a_k - (q_1^T a_k)q_1 - \cdots - \left(q_{k-1}^T a_k\right)q_{k-1} \rightarrow q_k = \frac{\widetilde{q_k}}{\|\widetilde{q_k}\|}$

Gram–Schmidt (orthogonalization) algorithm

Example

Find orthogonal set for $a = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathrm{c} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Gram–Schmidt (orthogonalization) algorithm

- Why $\{q_1, q_2, \ldots, q_k\}$ is a orthonormal basis for span $\{a_1, a_2, \ldots, a_k\}$?

  - $\{q_1, q_2, \ldots, q_k\}$ are normalized.
  - $\{q_1, q_2, \ldots, q_k\}$ is a orthogonal set
  - a_i is a linear combination of $\{q_1, q_2, \ldots, q_i\}$

  span$\{q_1, q_2, \ldots, q_k\}$ = span$\{a_1, a_2, \ldots, a_k\}$

- q_i is a linear combination of $\{a_1, a_2, \ldots, a_i\}$

Gram–Schmidt (orthogonalization) algorithm

- Given n-vectors $a_1, \dots, a_k$ for $i = 1, \dots, k$

  1. Orthogonalization: $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - (q_{i-1}^T a_i)q_{i-1}$
  2. Test for linear dependence: if $\tilde{q}_i = 0$, quit
  3. Normalization: $q_i = \frac{\tilde{q}_i}{\|\tilde{q}_i\|}$

Note

- If G–S does not stop early (in step 2), $a_1, \dots, a_k$ are linearly independent.
- If G–S stops early in iteration $i = j$, then a_j is a linear combination of $a_1, \dots, a_{j-1}$ (so $a_1, \dots, a_k$ are linearly dependent)

$$a_j = (q_1^T a_j)q_1 + \cdots + (q_{j-1}^T a_j)q_{j-1}$$

Complexity of Gram–Schmidt algorithm

- Gram-Schmidt algorithm gives us an explicit method for determining if a list of vectors is linearly dependent or independent.

- What is complexity and number of flops for this algorithm?
  - $O(nk^2)$ why?
- Given n-vectors $a_1, \dots, a_k$ for $i = 1, \dots, k$

1. Orthogonalization: $\tilde{q}_i = a_i - (q_1^T a_i)q_1 - \cdots - \left(q_{i-1}^T a_i\right)q_{i-1}$

2. Test for linear dependence: if $\tilde{q}_i = 0$, quit

3. Normalization: $q_i = \frac{\tilde{q}_i}{\|\tilde{q}_i\|}$

Complexity of the Gram–Schmidt algorithm. We now derive an operation count for the Gram–Schmidt algorithm. In the first step of iteration i of the algorithm, $i-1$ inner products

$$q_1^T a_i, \quad \ldots, \quad q_{i-1}^T a_i$$

between vectors of length n are computed. This takes $(i-1)(2n-1)$ flops. We then use these inner products as the coefficients in $i-1$ scalar multiplications with the vectors $q_1, \ldots, q_{i-1}$. This requires $n(i-1)$ flops. We then subtract the $i-1$ resulting vectors from a_i, which requires another $n(i-1)$ flops. The total flop count for step 1 is

$$(i-1)(2n-1) + n(i-1) + n(i-1) = (4n-1)(i-1)$$

flops. In step 3 we compute the norm of $\tilde{q}_i$, which takes approximately $2n$ flops. We then divide $\tilde{q}_i$ by its norm, which requires n scalar divisions. So the total flop count for the ith iteration is $(4n-1)(i-1) + 3n$ flops.

The total flop count for all k iterations of the algorithm is obtained by summing our counts for $i = 1, \ldots, k$:

$$\sum_{i=1}^{k}((4n-1)(i-1) + 3n) = (4n-1)\frac{k(k-1)}{2} + 3nk \approx 2nk^2,$$

where we use the fact that

$$\sum_{i=1}^{k}(i-1) = 1 + 2 + \cdots + (k-2) + (k-1) = \frac{k(k-1)}{2}, \qquad (5.7)$$

which we justify below. The complexity of the Gram–Schmidt algorithm is $2nk^2$; its order is nk^2. We can guess that its running time grows linearly with the lengths

Orthonormal basis

Corollary

Every finite-dimensional inner product space has an orthonormal basis.

Conclusion

Existence of Orthonormal Bases

- Every finite-dimensional inner product space has an orthonormal basis.
- Since finite-dimensional inner product spaces (by definition) have a basis consisting of finitely many vectors, and the Gram-Schmidt process tells us how to convert that basis into an orthonormal basis, we now know that every finite-dimensional inner product space has an orthonormal basis.

References

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- Linear Algebra and Optimization for Machine Learning
- Introduction to Applied Linear Algebra Vectors, Matrices, and Least Squares