Centrality

CE642: Social and Economic Networks
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01

Centrality

Why a centrality measure?

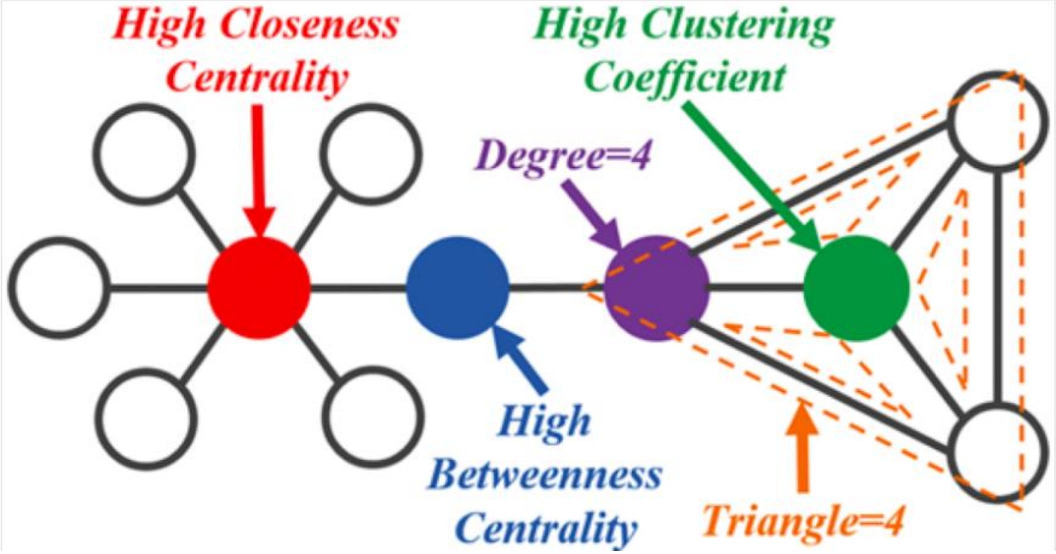
- Often, we are interested in identifying IMPORTANT network components
  - Nodes
  - Edges
- Central components may play critical role in network functions
  - Robustness
  - Collective behavior
  - Synchronization
  - Information spreading
  - Social dynamics
  - …

Centrality

- Which nodes are most 'central'?
- Definition of 'central' varies by context/purpose.
- Local measure:
  - Degree
- Relative to the rest of the network:
  - Closeness
  - Betweenness
  - Eigenvector (Bonacich power centrality)
- How evenly is centrality distributed among nodes?
  - Centralization
  - …

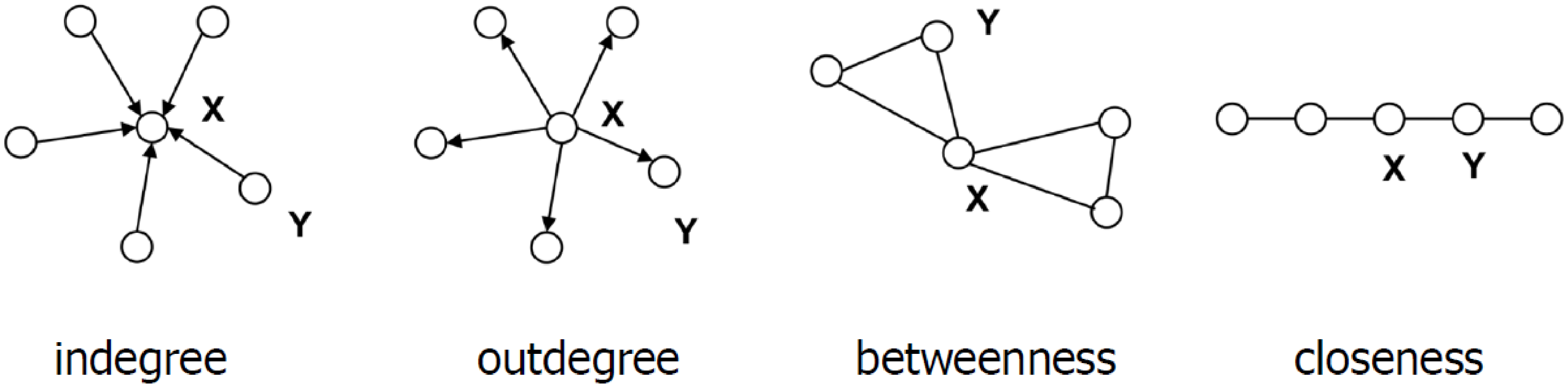
Network Centrality

- Given a social network, which nodes are more **important** or **influential**?
- **Centrality measures** were proposed to account for the importance of the nodes of a network



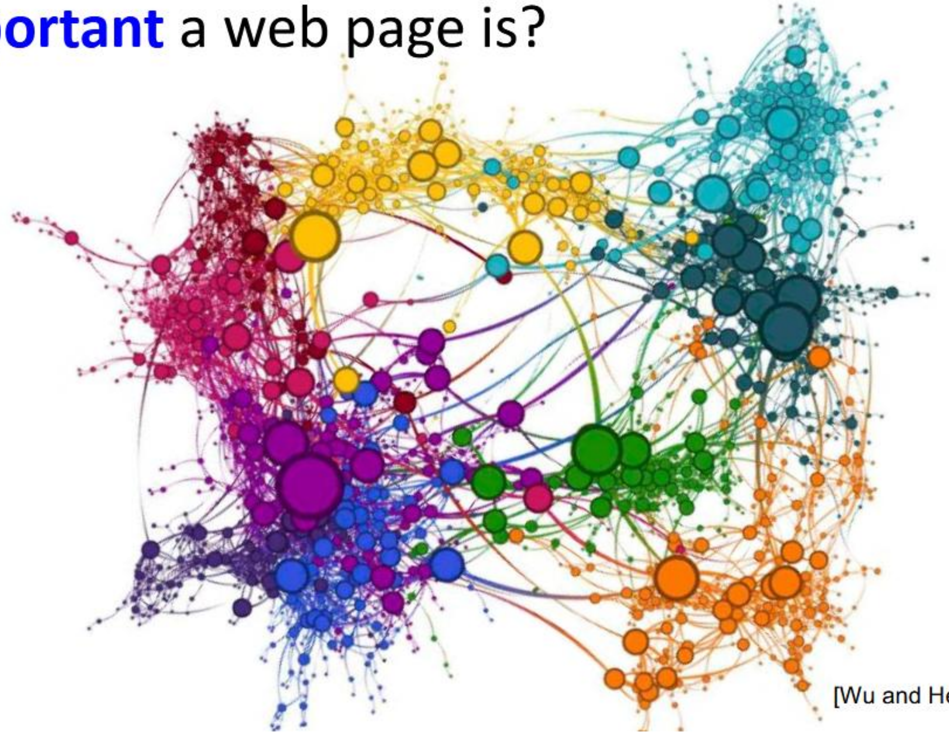
Network Centrality

- In each of the following networks, X has higher centrality than Y according to a particular measure



Network Centrality

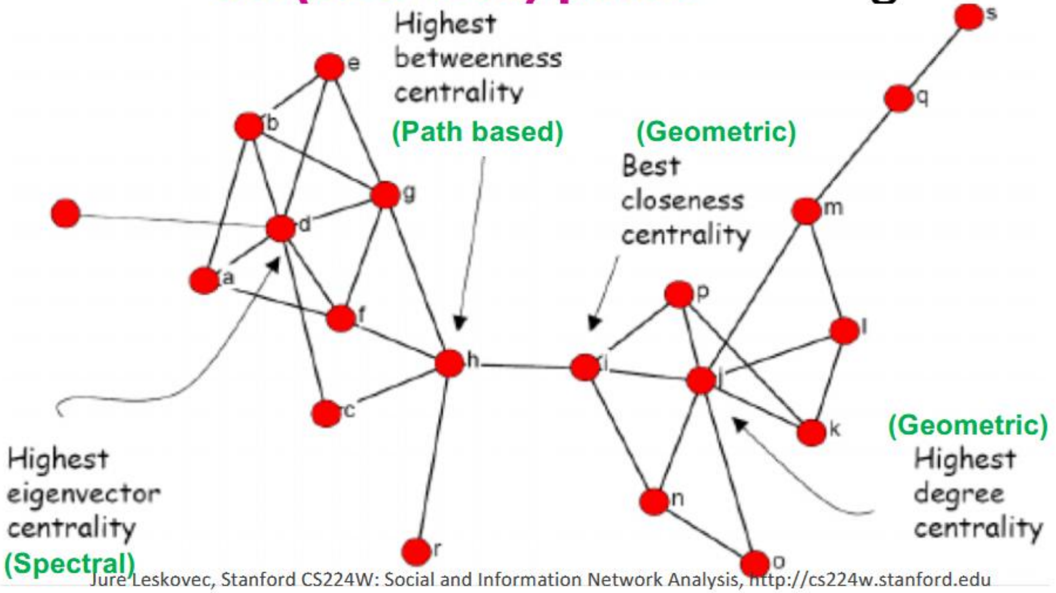
- **Centrality** is used often for detecting:
  - How **influential** a person is in a social network?
  - How **well used** a road is in a transportation network?
  - How **important** a web page is?



[Wu and He'15]

Centrality Measures

- **Geometric Measures**:
  - Importance is a **function of distances** to other nodes.
- **Spectral Measures**:
  - Based on the **eigen-structure** of some graph-related matrix
- **Path-based Measures**:
  - Take into account **all (shortest) paths** coming into a node



Jure Leskovec, Stanford CS224W: Social and Information Network Analysis, http://cs224w.stanford.edu 4

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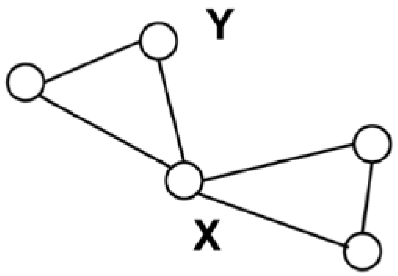
Path-based Measures for Centrality

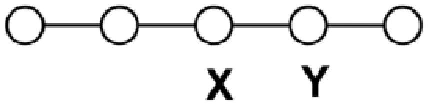
Betweenness Centrality

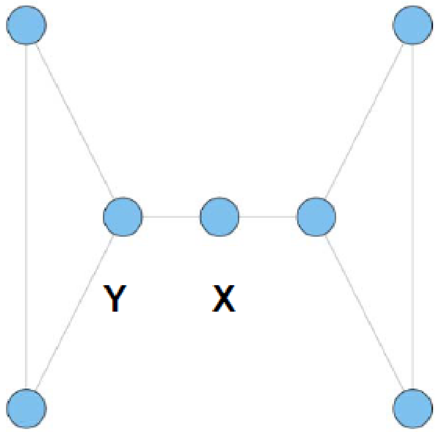
- Measure of centrality in a graph based on shortest paths:
  - Edge Betweenness Centrality
  - Node Betweenness Centrality
- In a telecommunications network, a node with higher betweenness centrality would have more control over the network, because more information will pass through that node.

Betweenness Centrality

- Intuition: how many pairs of individuals would have to go through you in order to reach one another in the minimum number of hops?
- Who has higher betweenness, X or Y?







Edge Betweenness Centrality

$$\rho_{ij} = \sum_{p \neq q} \left(\Gamma_{pq}\left(e_{ij}\right) / \Gamma_{pq} \right)$$

Γ_{pq} is the number of shortest paths from the *p*-th to the *q*-th node

$\Gamma_{pq}(e_{ij})$ is the number of these paths making use of e_{ij}.

- **Usually the betweenness is normalized by** $\boxed{/\,[(n-1)(n-2)/2]}$

Number of possible edges

Node Betweenness Centrality

$$C_i = \sum_{p \neq i \neq q} \left(\Gamma_{pq}(i) / \Gamma_{pq} \right)$$

Γ_{pq} is the number of shortest paths from the *p*-th to the *q*-th node

$\Gamma_{pq}(i)$ is the number of these shortest paths making use of the *i*-th node (except those that are start or end nodes is *i*).

- **Usually the betweenness is normalized by** $/[(n-1)(n-2)/2]$

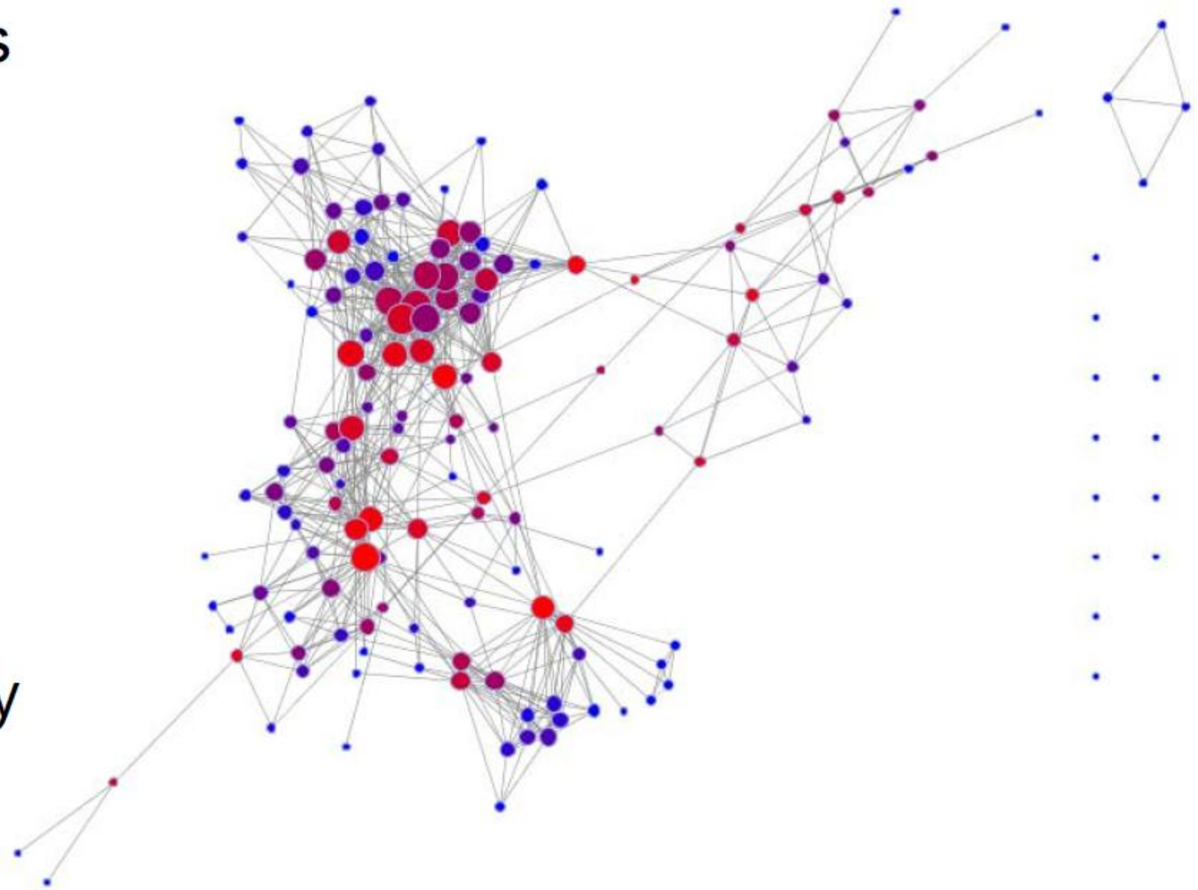
number of pairs of vertices excluding the vertex itself

Betweenness centrality: an example

Nodes are sized by degree, and colored by betweenness.

Can you spot nodes with high betweenness but relatively low degree? Explain how this might arise.

What about high degree but relatively low betweenness?

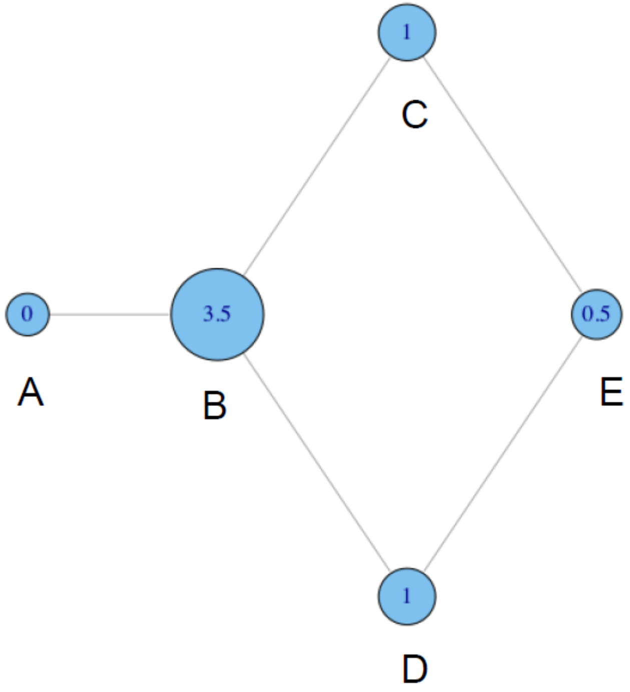


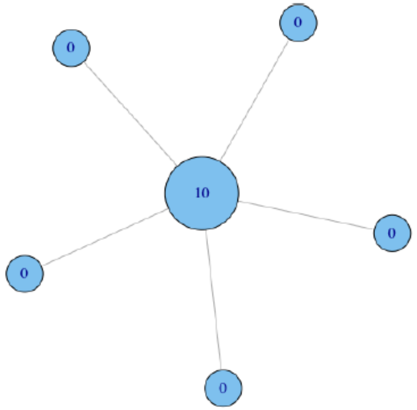
Betweenness Centrality

Why do C and D each have betweenness 1?

They are both on shortest paths for pairs (A,E), and (B,E), and so must share credit: $\frac{1}{2}+\frac{1}{2} = 1$

Can you figure out why B has betweenness 3.5 while E has betweenness 0.5?





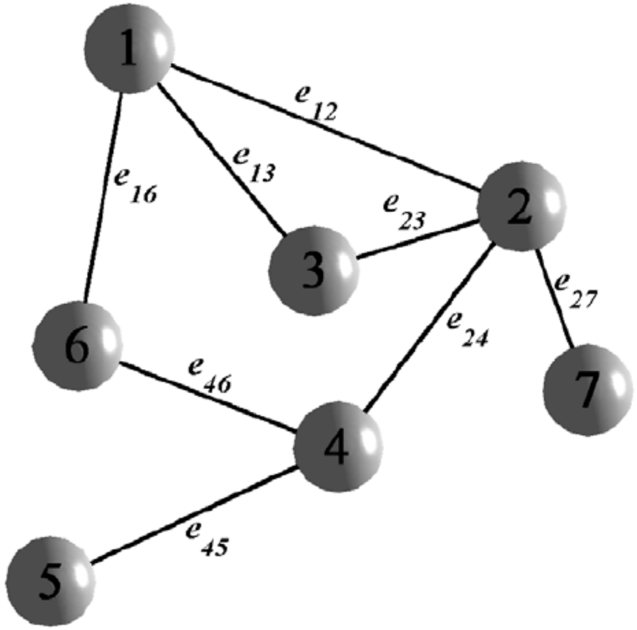
Connection Graph Stability Scores

- In some applications the importance of the shortest paths are also important
- The importance of the shortest paths might be different
- The more important paths making use of an edge the more its importance
- A simple measure of importance would be the length
- The connection graph stability (CGS) method takes into account this issue
- It has application in synchronization analysis
- The CGS-score b_{ij} for the link between the nodes i and j is defined as

$$b_{ij} = \sum_{u=1}^{n-1} \sum_{v>u;e_{ij} \in P_{uv}}^{n} |P_{uv}|$$

- $|P_{uv}|$: length of path P_{uv} between the nodes u and v

Connection Graph Stability Scores



P_{12} = e_{12}, P_{13} = e_{13}, P_{14} = $e_{12}e_{24}$,
P_{15} = $e_{16}e_{46}e_{45}$, P_{16} = e_{16}, P_{17} = $e_{12}e_{77}$,
P_{23} = e_{23}, P_{24} = e_{24}, P_{25} = $e_{24}e_{45}$,
P_{26} = $e_{24}e_{46}$, P_{27} = e_{27}, P_{34} = $e_{23}e_{24}$,
P_{35} = $e_{23}e_{24}e_{45}$, P_{36} = $e_{13}e_{16}$, P_{37} = $e_{23}e_{27}$,
P_{45} = e_{45}, P_{46} = e_{46}, P_{47} = $e_{24}e_{27}$, P_{56} =
$e_{45}e_{46}$, P_{57} = $e_{45}e_{24}e_{27}$, P_{67} = $e_{16}e_{12}e_{27}$

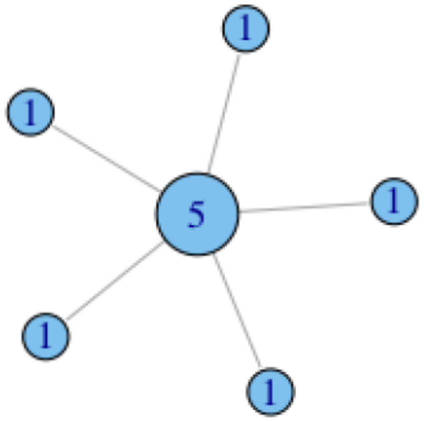
$$b_{12} = |P_{12}| + |P_{14}| + |P_{17}| + |P_{67}| = 1 + 2 + 2 + 3 = 8,$$
$$b_{13} = |P_{13}| + |P_{36}| = 1 + 2 = 3,$$
$$b_{16} = |P_{15}| + |P_{16}| + |P_{36}| + |P_{67}| = 3 + 1 + 2 + 3 = 9,$$
$$b_{23} = |P_{23}| + |P_{34}| + |P_{35}| + |P_{37}| = 1 + 2 + 3 + 2 = 8,$$
$$b_{24} = |P_{14}| + |P_{24}| + |P_{25}| + |P_{26}| + |P_{34}| + |P_{35}| + |P_{47}|$$
$$+ |P_{57}| = 2 + 1 + 2 + 2 + 2 + 3 + 2 + 3 = 17,$$
$$b_{27} = |P_{17}| + |P_{27}| + |P_{37}| + |P_{47}| + |P_{57}| + |P_{67}| =$$
$$2 + 1 + 2 + 2 + 3 + 3 = 13,$$
$$b_{45} = |P_{15}| + |P_{25}| + |P_{35}| + |P_{45}| + |P_{56}| + |P_{57}| =$$
$$3 + 2 + 3 + 1 + 2 + 3 = 14,$$
$$b_{46} = |P_{15}| + |P_{26}| + |P_{46}| + |P_{56}| = 3 + 2 + 1 + 2 = 8.$$

03

Geometric Measures for Centrality

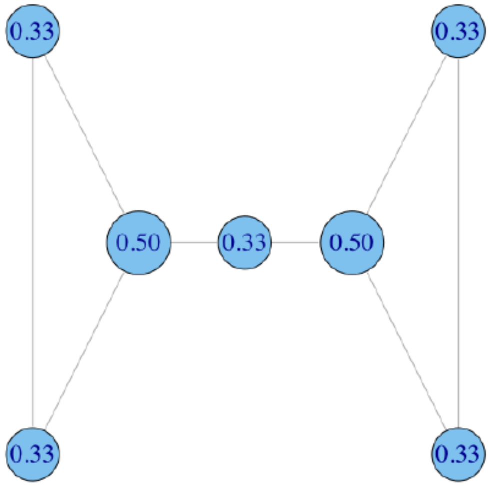
Degree centrality (undirected)

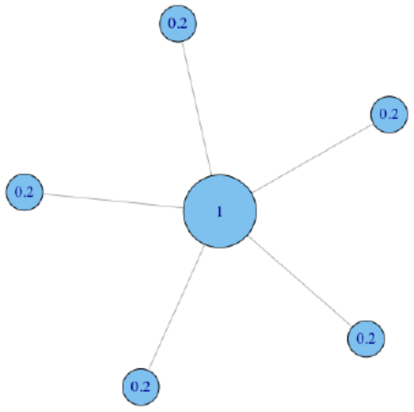
The more the friends the more the importance (the richer the better)



When is the number of connections the best centrality measure?

- people who will do favors for you
- people you can talk to / have a drink with





Normalized degree centrality:

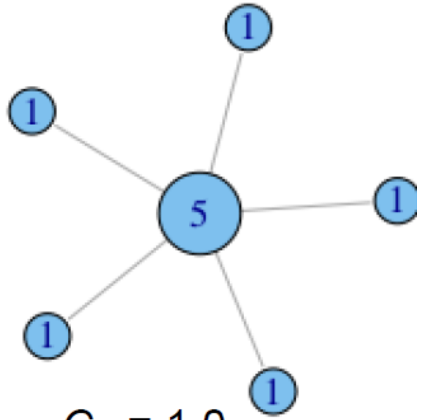
Degree is divided by the max. possible, i.e. (N-1)

How equal are the nodes?

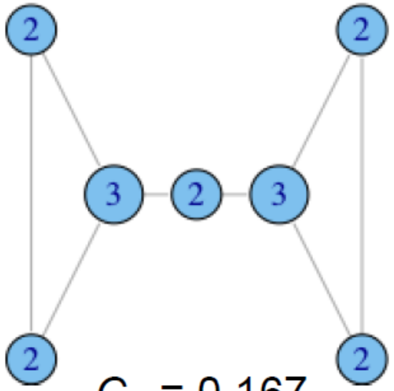
- How much variation is there in the centrality scores among the nodes?
- Freeman's general formula for centralization (can use other metrics, e.g. Gini coefficient or standard deviation):

maximum value in the network

$$C_D = \frac{\sum_{i=1}^{g}\left[\boxed{C_D(n^*)} - C_D(i)\right]}{[(N-1)(N-2)]}$$



$C_D = 1.0$



$C_D = 0.167$

$C_D = 0.167$

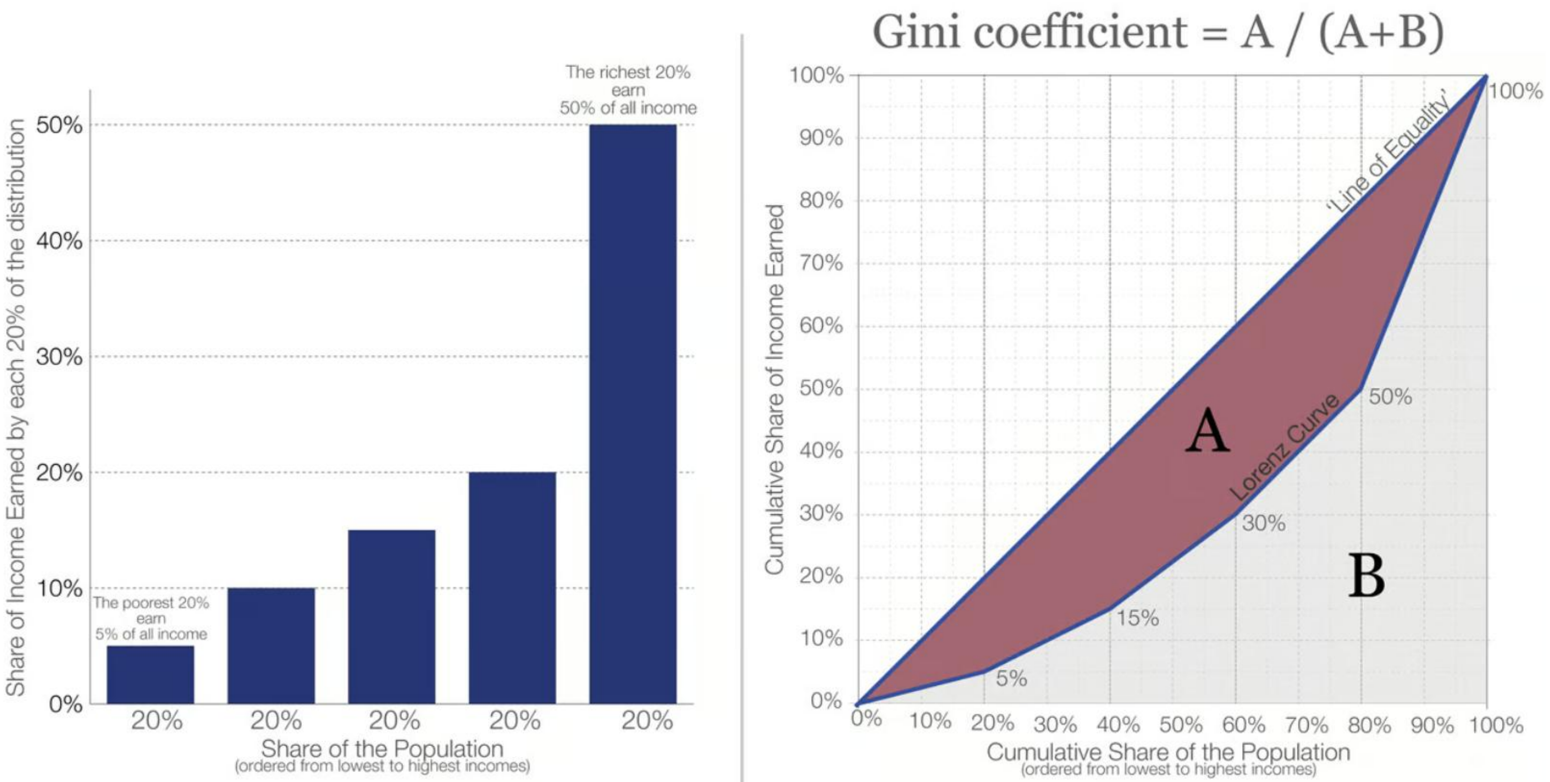
Gini Coefficient (Index)

The bar chart on the left shows a simple distribution of incomes. The total population is split up in 5 parts and ordered from the poorest to the richest 20%. The bar chart shows how much income each 20% part of of the income distribution earns.

The chart on the right shows the same information in a different way, both axis show the cumulative shares:
The poorest 20% of the population earn 5% of the total income, the next 20% earn 10% – so that the poorest 40% of the population earn 15% etc.
The curve resulting from this way of displaying the data is called the Lorenz Curve.

If there was no income inequality the resulting Lorenz Curve would be a straight line – the 'Line of Equality'.
A larger area (A) between the Lorenz Curve and the Line of Equality means a higher level of inequality.

The ratio of A/(A+B) is therefore a measure of inequality and is referred to as the Gini coefficient, Gini index, or simply the Gini.



Gini Coefficient (Index)

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Information Sciences

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Sparsity measure of a network graph: Gini index

Swati Goswami [a] [b] [1], C.A. Murthy [a], Asit K. Das [b]

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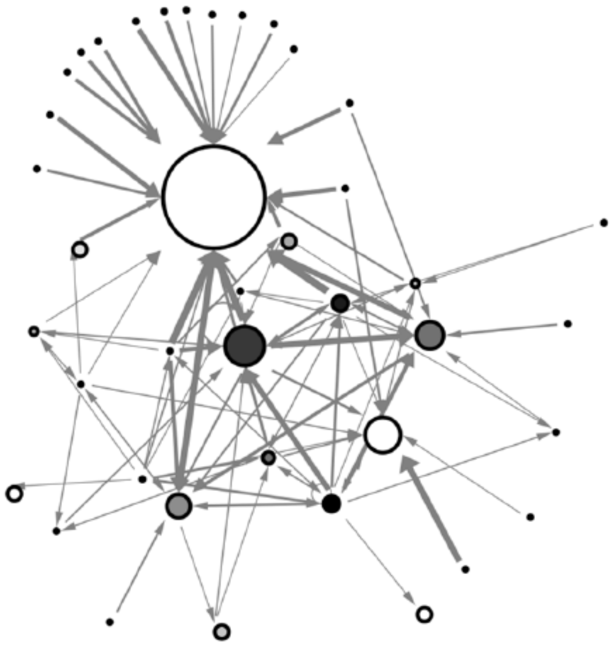
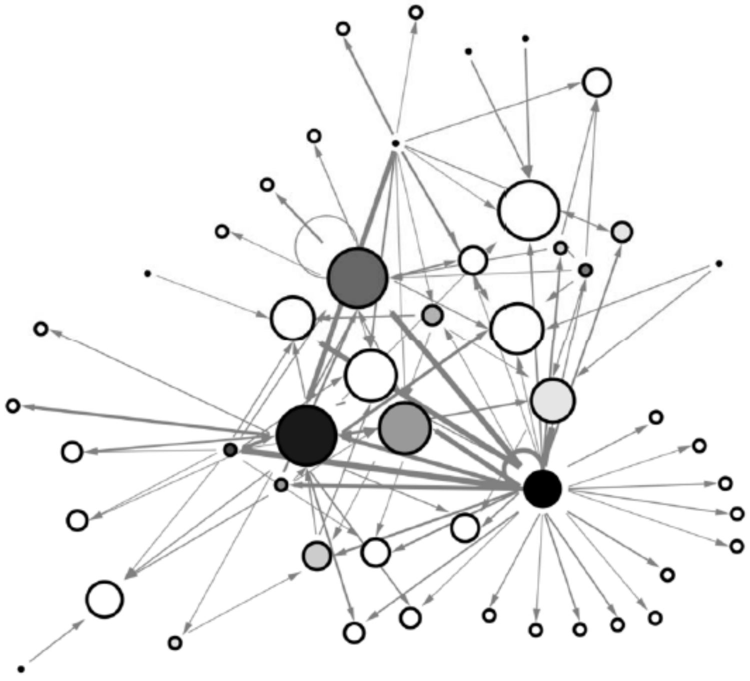
Example: financial trading networks



high centralization: one node trading with many others



low centralization: trades are more evenly distributed

Characteristic path length

- A network with N nodes
  - Compute the shortest path (distance) between any two nodes d_{ij}
  - The length of the path is the number of edges (unweighted networks) or the weighted sum of the edges (weighted networks)
  - If the nodes are not connected, the path length between them is set to infinity
  - It is also called average geodesic distance
  - If dij is infinity, it diverges
  - Many times we compute the average only over the connected pairs of nodes (that is, we ignore “infinite” length paths)
  - $l = \frac{1}{N(N-1)} \sum_{i,j,i \neq j} d_{ij}$

Efficiency

- In this way the divergence is avoided
- The inverse of efficiency E is called harmonic mean
- Efficiency is an indicator of the traffic capacity of the
- network
- The couple of disconnected nodes have a contribution of zero in computing E
- The more the values of E are the more the communication-efficient the network is
- It is also called global efficiency of the network.

$$E = \frac{1}{N(N-1)} \sum_{i,j,i \neq j} \frac{1}{d_{ij}}$$

Efficiency

- Higher Efficiency → Faster communication, better connectivity.
- Why is Efficiency Important?
  - 🚀 High Efficiency → Network is Well-Connected
    - Information spreads quickly.
    - Fewer intermediate steps needed.
    - Helps in optimizing transportation, communication, and social interactions.
  - 🐢 Low Efficiency → Poor Connectivity
    - Long paths between nodes.
    - Slower communication and bottlenecks.
    - Less effective in handling information flow.

$$E = \frac{1}{N(N-1)} \sum_{i,j,i \neq j} \frac{1}{d_{ij}}$$

The Role of Connectivity in Efficiency

Network Type	Effect on E
Fully Connected (Strongly Connected Component - SCC)	🔺 High E, every node can reach any other node efficiently.
Weakly Connected Component (WCC)	🔻 Lower E, some nodes may only be accessible in one direction.
Disconnected Components	⛔ Very Low E, as some distances are infinite (ignored in practical calculations).

The Power of Shortcuts

- A shortcut in a graph is an additional edge that significantly reduces the shortest path distance between two nodes without being essential for the overall connectivity of the network.
- Extra direct connections between distant nodes.
- Reduce the shortest path distance d_{ij}.
- Improve efficiency without requiring full connectivity.
- Real-World Examples:
  - Social Networks → Influencers create bridges between distant groups.
  - Transport Networks → Highways and express routes reduce travel time.
  - Computer Networks → Fast routing through backbone connections (CDNs).
- Fewer hops → Shorter paths → Higher Efficiency.

Vulnerability

- It is important to know which component (nodes or edges) are crucial to the best performance.
- The more the drop in the efficiency by removing a component the more crucial that component.
- Degree (hub node) might be a criterion
  - Only degree is not enough, e.g. all vertices of a binary tree network have equal degree, i.e. no hub, but disconnection of vertices closer to the root and the root itself have a greater impact than of those near the leaves.
- The amount of change in the efficiency (or other network properties) as a component is removed can be an indicator of the vulnerability

Vulnerability

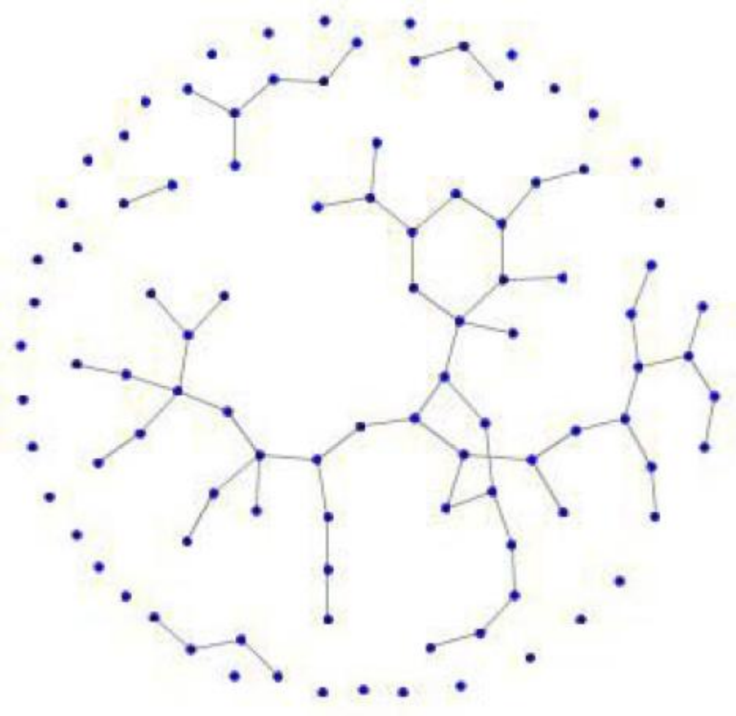
$$V_i = \frac{E - E_i}{E} \qquad V = \max_i V_i$$

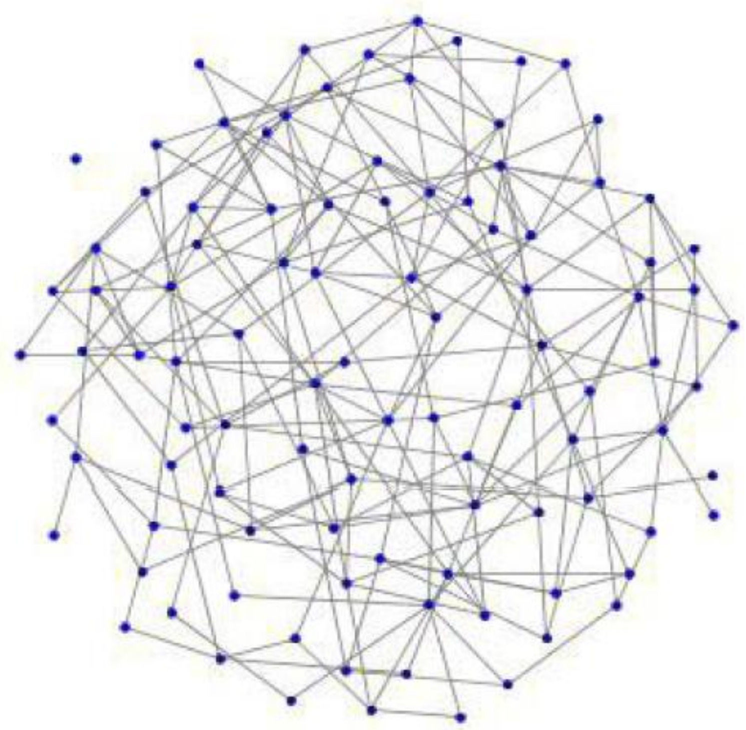
- where V_i is the vulnerability of component i and E_i is the of i E efficiency the networks by removed that component.

- V can be regarded as the vulnerability of the network the ordered distribution of nodes with respect to their vulnerability V_i is related to the network hierarchy.
- The most vulnerable (critical) node occupies the highest position in the network hierarchy.
- The same is also true for the edges.

Density

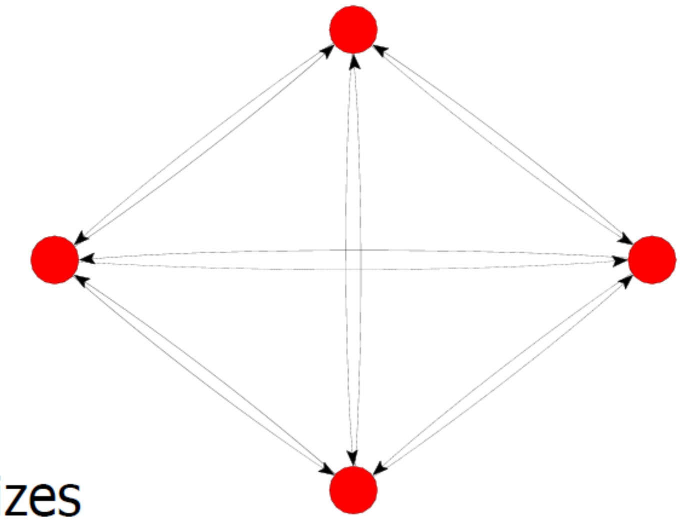
- How dense the networks are?





Density

- Number of the connections that may exist between n nodes
  - directed graph
    $e_{max} = n*(n-1)$
    each of the n nodes can connect to (n-1) other nodes
  - undirected graph
    $e_{max} = n*(n-1)/2$
    since edges are undirected, count each one only once
- What fraction are present?
  - density = e/ e_{max}
  - For example, out of 12 possible connections, this graph has 7, giving it a density of 7/12 = 0.583



- Would this measure be useful for comparing networks of different sizes (different numbers of nodes)?

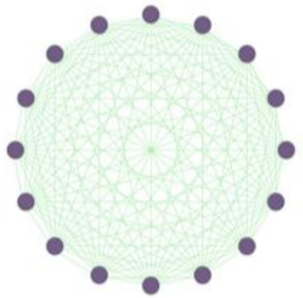
Density

— for *undirected* G *density* ρ is defined as

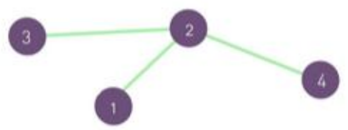
$$\rho = \frac{2m}{n(n-1)} = \frac{\langle k \rangle}{n-1}$$

— for *directed* G *density* $\rho^{\cdot}$ is defined as

$$\rho^{\cdot} = \frac{m}{n(n-1)} = \frac{\langle k^{\cdot} \rangle}{n-1}$$



complete $m = \binom{n}{2}$



tree $m = n - 1$

— G is *dense* if $\rho \to \text{const.}$ as $n \to \infty$ thus $\langle k \rangle = \mathcal{O}(n)$

— G is *sparse* if $\rho \to 0$ as $n \to \infty$ thus $\langle k \rangle \neq \mathcal{O}(n)$

Closeness

- What if it's not so important to have many direct friends?
  - Degree Centrality is not important
- Or be "between" others
  - Betweeness Centrality is not important
- But one still wants to be in the "middle" of things, not too far from the center.
- Closeness is based on the length of the average shortest path between a node and all other nodes in the network

Example

- **High Degree Centrality**
  - A person who has many direct friends in a social network.
- **High Betweenness Centrality**
  - A person who acts as a bridge between two separate groups.
- **High Closeness Centrality**
  - A person who can reach anyone in the network with the fewest intermediaries, even if they don’t have many direct friends.
- Closeness Centrality focuses on being "near" other nodes in terms of short paths across the entire network, rather than just having many direct connections!

Closeness

- Formula:

Closeness Centrality:

$$C_c(i) = \left[\sum_{j=1}^{N} d(i,j)\right]^{-1}$$

Normalized Closeness Centrality

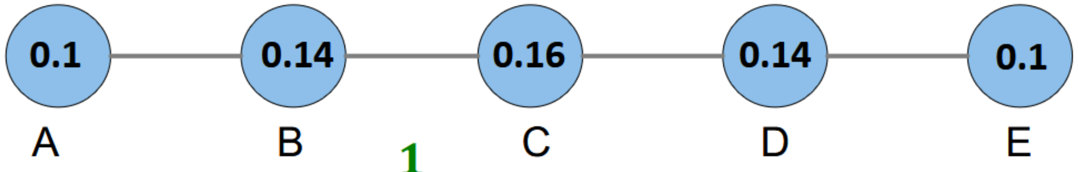
$$C'_C(i) = \left[\sum_{j=1}^{N} d(i,j)/(N-1)\right]^{-1}$$

  - **Closeness Centrality:**

$$c_{\text{clos}}(x) = \frac{1}{\sum_y \boxed{d(y,x)}}$$

length of the shortest path from x to y

  - How much a vertex can communicate without relying on third parties for his messages to be delivered

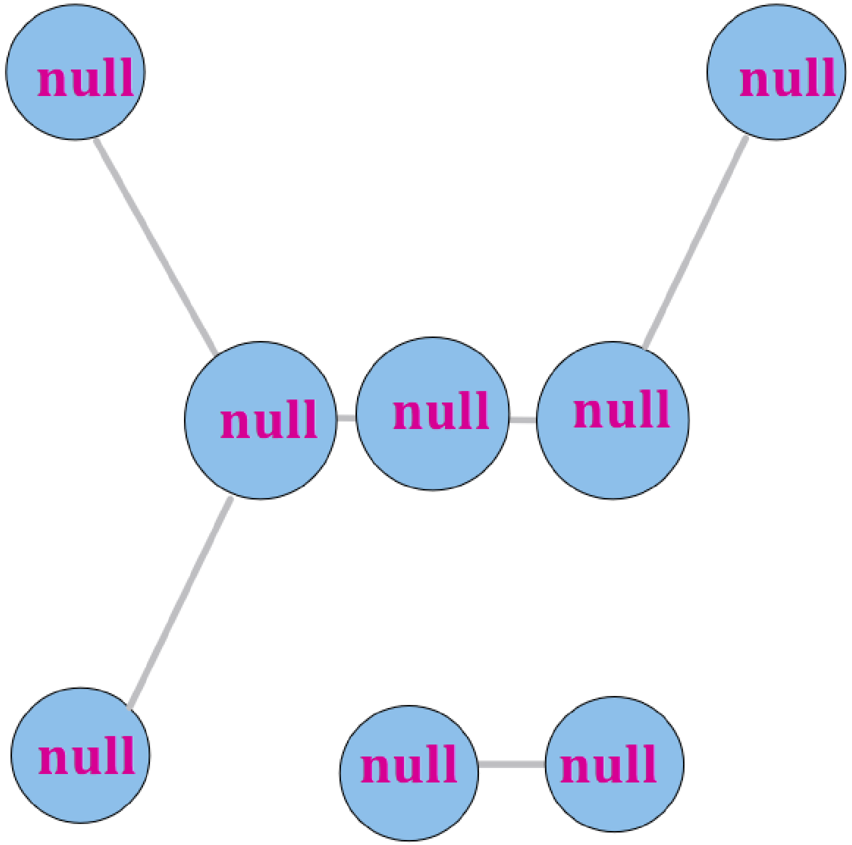


$$c_{clos}(A) = \frac{1}{1+2+3+4} = 0.1$$

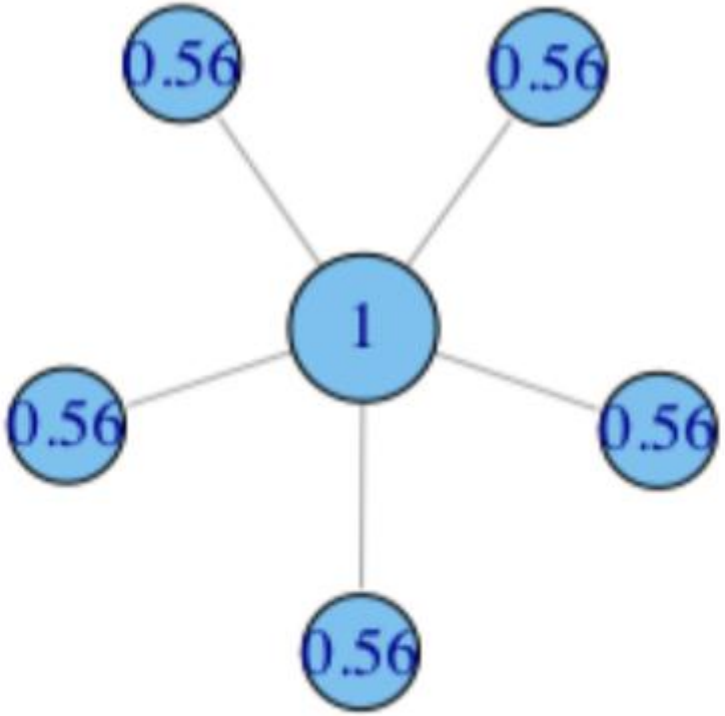
$$C'_c(A) = \left[\frac{\sum_{j=1}^{N} d(A,j)}{N-1}\right]^{-1} = \left[\frac{1+2+3+4}{4}\right]^{-1} = \left[\frac{10}{4}\right]^{-1} = 0.4$$

- **Problem**: The graph must be **(strongly) connected!**

Closeness Example



We get null score for all nodes,
if the graph is not connected!



Harmonic Centrality

- **Geometric measures**
  - **Harmonic Centrality**:
    - Replace the average distance with the harmonic mean of all distances.
    - The $n(n-1)$ distances between every pair of distinct nodes:

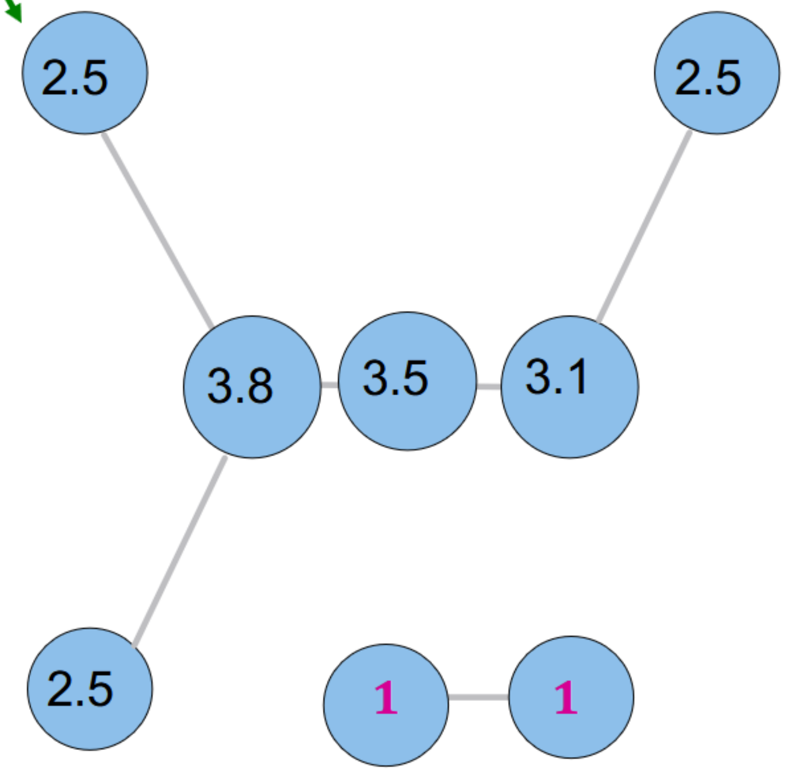
$$c_{\text{har}}(x) = \boxed{\sum_{y \neq x} \frac{1}{d(y,x)}} = \sum_{d(y,x)<\infty, y \neq x} \frac{1}{d(y,x)}$$

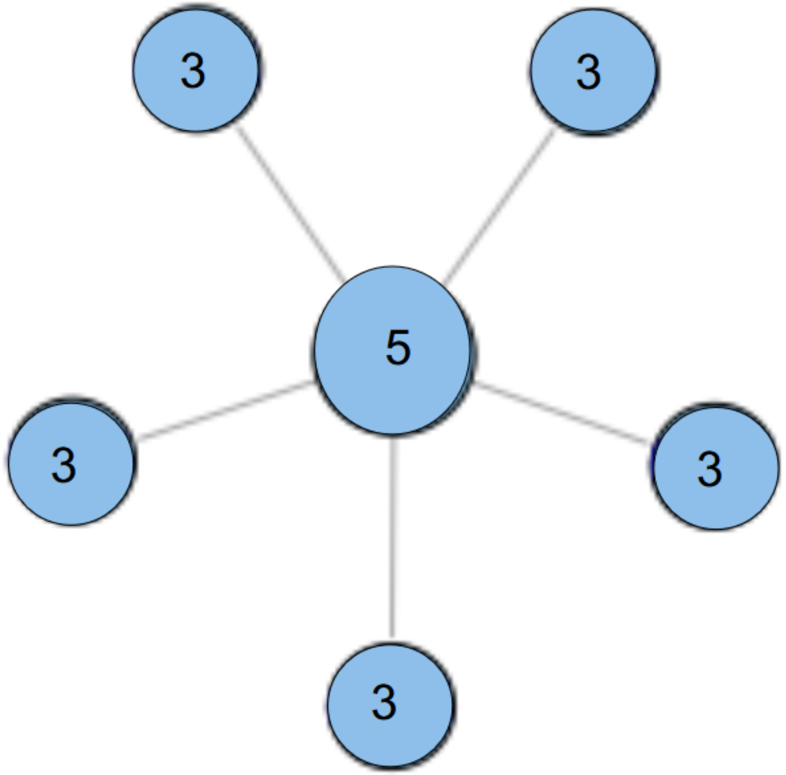
Harmonic mean

    - Strongly correlated to closeness centrality
    - Naturally also accounts for nodes y that cannot reach x
    - Can be applied to graphs that are **not strongly connected**

Harmonic Centrality Example

$$c_{harm} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = 2.5$$

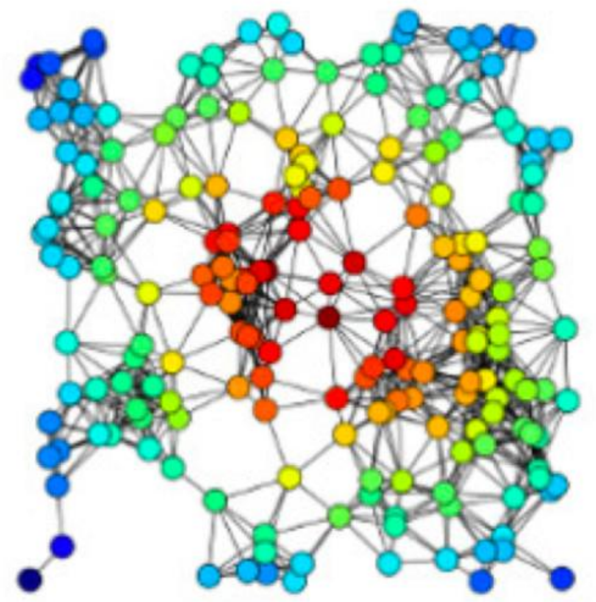




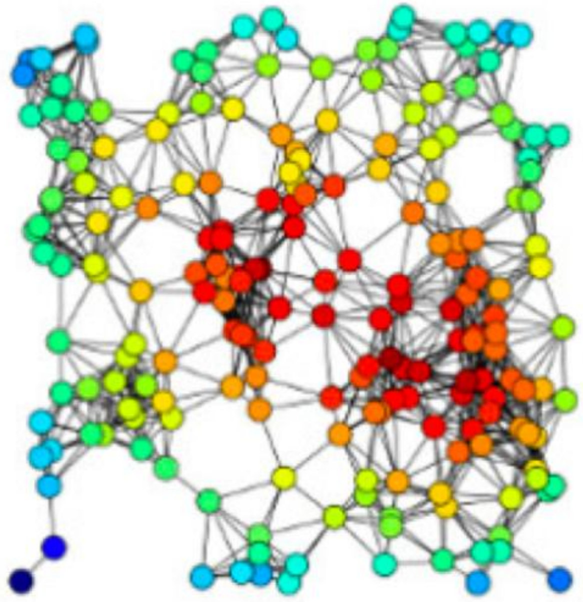
Comparison

- Closeness Centrality is affected by average distances, while Harmonic Centrality is influenced by nearby nodes.
- A node can be well-positioned (high Closeness) but still have many distant nodes that lower its Harmonic score.
- Harmonic Centrality gives more weight to close neighbors, whereas Closeness considers all distances equally.

Closeness vs Harmonic Centrality



Red nodes are closer to all the other nodes



Red nodes are closer to all the other nodes, and have larger degrees

Examples of Closeness centrality, and Harmonic Centrality of the same graph.

Let's Think

- Can a Node Have High Harmonic Centrality but Low Degree?
  - Imagine you have only two friends, but they are well-connected influencers in the network. Your degree is low (only 2 connections). However, because your friends have strong connections, you can quickly reach many people.
- Can a Node Have High Degree but Low Harmonic Centrality?
  - Imagine a node has 10 direct connections, but all these connections are to each other and not to the rest of the network. Degree is high (10 connections), but reaching other parts of the network requires multiple hops.
- Can a Node Have High Closeness but Low Harmonic Centrality?
  - Consider the tree or ring networks.
- Can a Node Have Low Closeness but High Harmonic Centrality?
  - In a tree structure, a node close to a highly connected hub can still have low Closeness (because reaching deep parts of the tree takes many steps). However, its Harmonic Centrality is high because it can reach nearby nodes very efficiently.

04

Spectral Measures for Centrality

Spectral Measures

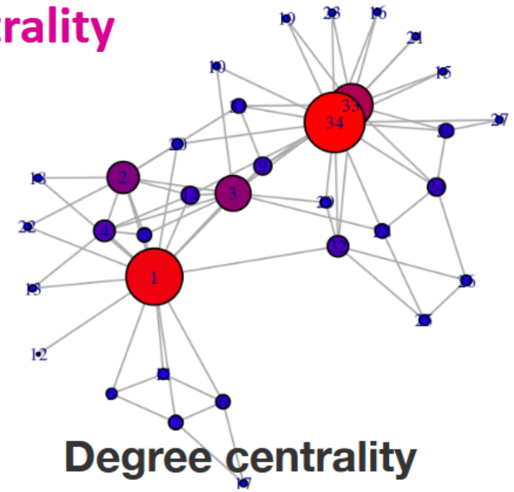
- **Spectral measures**
  - Compute the left dominant eigenvector of some matrix derived from the graph
  - **Idea:** A node's centrality is a function of the **centrality of its neighbors**
    - Nodes connected to central nodes has a larger centrality score than those connected to non-central nodes.
    - Eigenvector Centrality
    - Katz's Index
    - Page Rank
    - Hits

Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix, defined as:

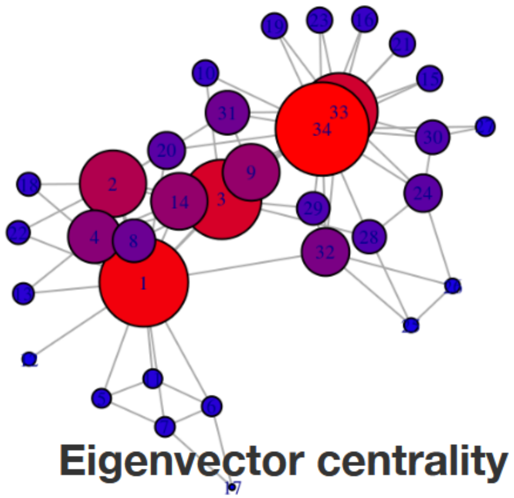
$$A_{ij} = 1 \quad \text{if there is an edge from node } j \text{ to node } i$$

This convention means:

- Column j tells you where node j is pointing (i.e., outgoing edges from j),
- Row i shows you all incoming edges to node i.



Degree centrality



Eigenvector centrality

Review

Left and Right Eigenvectors

We've been working with right eigenvectors and right eigenvalues:

$$A\vec{v}_d = \lambda_d \vec{v}_d$$

There may also be left eigenvectors, which are row vectors $\vec{u}_d$ and corresponding left eigenvalues κ_d:

$$\vec{u}_d^T A = \kappa_d \vec{u}_d^T$$

Eigenvectors on both sides of the matrix

You can do an interesting thing if you multiply the matrix by its eigenvectors both before and after:

$$\vec{u}_i^T(A\vec{v}_j) = \vec{u}_i^T(\lambda_j\vec{v}_j) = \lambda_j\vec{u}_i^T\vec{v}_j$$

. . . but. . .

$$(\vec{u}_i^T A)\vec{v}_j = (\kappa_i\vec{u}_i^T)\vec{v}_j = \kappa_i\vec{u}_i^T\vec{v}_j$$

There are only two ways that both of these things can be true. Either

$$\kappa_i = \lambda_j \quad \text{or} \quad \vec{u}_i^T\vec{v}_j = 0$$

Left and right eigenvectors must be paired!!

There are only two ways that both of these things can be true.
Either

$$\kappa_i = \lambda_j \quad \text{or} \quad \vec{u}_i^T \vec{v}_j = 0$$

Remember that eigenvalues solve $|A - \lambda_d I| = 0$. In almost all cases, the solutions are all distinct (A has distinct eigenvalues), i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$. That means there is **at most one** λ_j that can equal each κ_i:

$$\begin{cases} i \neq j & \vec{u}_i^T \vec{v}_j = 0 \\ i = j & \kappa_i = \lambda_i \end{cases}$$

Symmetric matrices: left=right

If A is symmetric ($A = A^T$), then the left and right eigenvectors and eigenvalues are the same, because

$$\lambda_i \vec{u}_i^T = \vec{u}_i^T A = (A^T \vec{u}_i)^T = (A\vec{u}_i)^T$$

... and that last term is equal to $\lambda_i \vec{u}_i^T$ if and only if $\vec{u}_i = \vec{v}_i$.

Dominant Eigenvector

- The eigenvector corresponded to dominant eigenvalue.

Definition

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvectors of an $n \times n$ matrix A. λ_1 is called the dominant eigen value of A if

$$|\lambda_1| \geq |\lambda_i| \; i = 2, \ldots, n$$

Stochastic Matrix

- A matrix M is called a stochastic matrix if all the entries are positive and the sum of the elements in each column is equal to 1.
- A stochastic matrix is also called probability matrix, transition matrix, substitution matrix, or Markov matrix.

- What is the right dominant eigenvalue of a row-based stochastic matrix?
  - 1 (proof in next 2 slides)

Theorem about Stochastic Matrix

The stochastic matrix A has an eigenvalue 1.

We compute that

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ a_{21} + a_{22} + \cdots + a_{2n} \\ \vdots \\ a_{n1} + a_{n2} \cdots + a_{nn} \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Here the second equality follows from the definition of a right stochastic matrix.

(Each row sums up to 1.)

This computation shows that 1 is an eigenvector of A and $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 1.

Theorem about Stochastic Matrix

The absolute value of any eigenvalue of the stochastic matrix A is less than or equal to 1.

Let λ be an eigenvalue of the stochastic matrix A and let $\mathbf{v}$ be a corresponding eigenvector.

That is, we have

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Comparing the i-th row of the both sides, we obtain

$$a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n = \lambda v_i \quad (*)$$

for $i = 1, \ldots, n$.

Theorem about Stochastic Matrix

Let

$$|v_k| = \max\{|v_1|, |v_2|, \dots, |v_n|\},$$

namely v_k is the entry of $\mathbf{v}$ that has the maximal absolute value.

Note that $|v_k| > 0$ since otherwise we have $\mathbf{v} = \mathbf{0}$ and this contradicts that an eigenvector is a nonzero vector. Then from (*) with $i = k$, we have

$$\begin{aligned} |\lambda| \cdot |v_k| &= |a_{k1}v_1 + a_{k2}v_2 + \cdots + a_{kn}v_n| \\ &\leq a_{k1}|v_1| + a_2|v_2| + \cdots + a_{kn}|v_n| && \text{(by the triangle inequality and } a_{ij} \geq 0) \\ &\leq a_{k1}|v_k| + a_2|v_k| + \cdots + a_{kn}|v_k| && \text{(since } |v_k| \text{ is maximal)} \\ &= (a_{k1} + a_{k2} + \cdots + a_{kn})|v_k| = |v_k|. \end{aligned}$$

Since $|v_k| > 0$, it follows that

$$\lambda \leq 1$$

as required.

Power Method

- The Power Method is a simple, iterative algorithm used to find the dominant eigenvector (the one associated with the largest eigenvalue) of a square matrix A.
- It's one of the easiest ways to compute an eigenvector without doing fancy matrix algebra.
- It works best when the matrix is large, sparse, and you only need the largest eigenvalue/vector.

Power Method

Given a square matrix $A \in \mathbb{R}^{n \times n}$, show that the **power method**:

$$\boldsymbol{x}^{(k)} = \frac{A\boldsymbol{x}^{(k-1)}}{\left|\left|A\boldsymbol{x}^{(k-1)}\right|\right|}$$

converges to the eigenvector corresponding to the **dominant eigenvalue** of A, as $k \to \infty$.

Power Method

- Step 1: Start with a random vector
  - Choose any non-zero vector (e.g., $[1, 1, \ldots, 1]^T$) as the initial guess.
- Step 2: Multiply by the matrix
  - Repeat: $x^{(k)} = A\, x^{(k-1)}$ to amplify the dominant eigenvector.
- Step 3: Normalize after each step
  - Rescale $x^{(k)} = x^{(k)} / \|x^{(k)}\|_2$ to keep values stable.
- Step 4: Repeat until convergence
  - Stop when $\|x^{(k)} - x^{(k-1)}\| < \varepsilon$ (e.g., $\varepsilon = 10^{-6}$).
  - Result: $x^{(k)} \approx$ dominant eigenvector.

Proof of Power Method

✅ Step 1: Decompose the initial vector

Since A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, we can write:

$$\mathbf{x}^{(0)} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Where $c_1 \neq 0$ (we assume it has a non-zero projection on the dominant eigenvector).

✅ Step 2: Multiply by A

Now apply A once:

$$\mathbf{x}^{(1)} = A\mathbf{x}^{(0)} = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_n A\mathbf{v}_n = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_n\lambda_n\mathbf{v}_n$$

Repeat this process k times:

$$\mathbf{x}^{(k)} = A^k\mathbf{x}^{(0)} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

Proof of Power Method

✅ Step 3: Factor out λ_1^k

$$\mathbf{x}^{(k)} = \lambda_1^k \left(c_1 \mathbf{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k \mathbf{v}_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^k \mathbf{v}_n \right)$$

Now observe:

Since $|\lambda_2/\lambda_1| < 1$, we have:

$$\left(\frac{\lambda_i}{\lambda_1} \right)^k \to 0 \quad \text{as } k \to \infty \quad \text{for all } i \geq 2$$

$$\lim_{k\to\infty} \frac{\mathbf{x}^{(k)}}{\|\mathbf{x}^{(k)}\|} = \pm \mathbf{v}_1$$

Assume that the eigenvector $\mathbf{v}_1$ is normalized to have unit length (i.e., $\|\mathbf{v}_1\| = 1$). Then the norm of $\mathbf{x}^{(k)}$ is approximately:

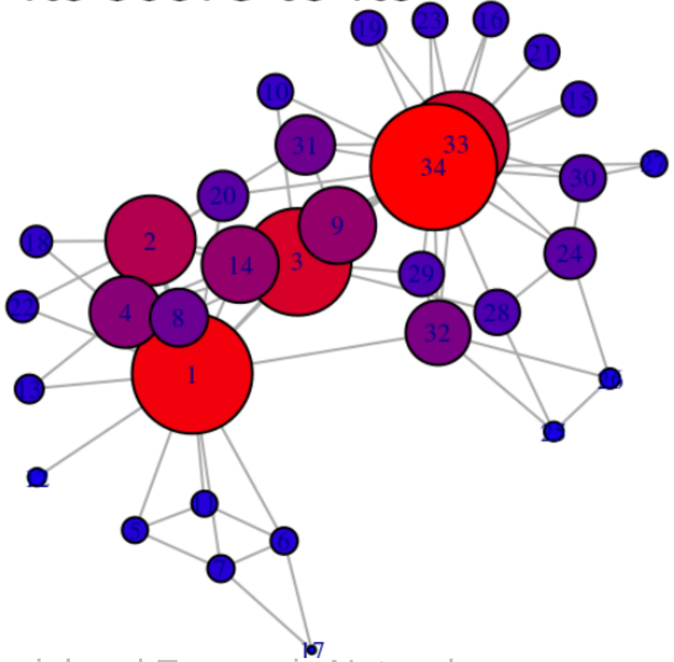
$$\|\mathbf{x}^{(k)}\| \approx \|c_1 \lambda_1^k \mathbf{v}_1\| = |c_1 \lambda_1^k| \cdot \|\mathbf{v}_1\| = |c_1 \lambda_1^k|$$

Eigencentrality

- **Spectral measures**
  - **Eigenvector Centrality**: Measure of the **influence** of a node in a network
  - **Idea:** Every node starts with the same score, and then each node gives away its score to its successors

$$c_{\mathrm{eig}}(x) = \frac{1}{\lambda} \sum_{y \to x} c_{\mathrm{eig}}(y)$$

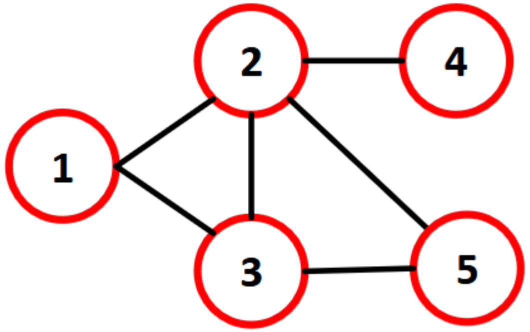
Normalization constant = $\|c_{eig}\|_2$



- **Intuitively**: Degree counts walks of length one, the eigenvalue centrality counts walks of length infinity

How to compute Eigencentrality?

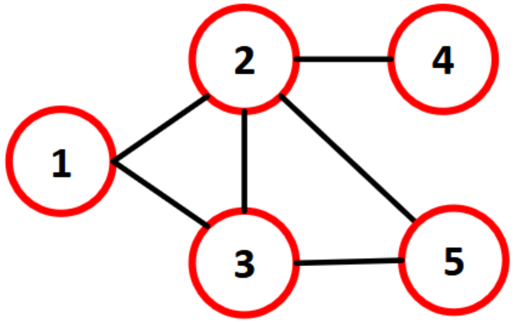
- **Power Iteration:**
  - Set $\boldsymbol{c}^{(0)} \leftarrow 1, k \leftarrow 1$
  - **1:** $\boldsymbol{c}^{(k)} \leftarrow \boldsymbol{A}\boldsymbol{c}^{(k-1)}$
  - **2:** $\boldsymbol{c}^{(k)} = \boldsymbol{c}^{(k)} / \left\|\boldsymbol{c}^{(k)}\right\|_2$
  - **3:** If $\left\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^{(k-1)}\right\| > \varepsilon$:
  - **4:** $\quad k \leftarrow k + 1$, goto **1**



$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \qquad c = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

How to compute Eigencentrality?

- **Power Iteration:**



Iteration 1

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \\ 2 \end{bmatrix} \equiv \begin{bmatrix} 0.34 \\ 0.68 \\ 0.51 \\ 0.17 \\ 0.34 \end{bmatrix}$$

$$\boldsymbol{A} \quad \boldsymbol{c}^{(0)} \quad \boldsymbol{c}^{(1)} = \boldsymbol{A}\boldsymbol{c}^{(0)} \quad \boldsymbol{c}^{(1)} = \boldsymbol{c}^{(1)} / \left\| \boldsymbol{c}^{(1)} \right\|_2$$

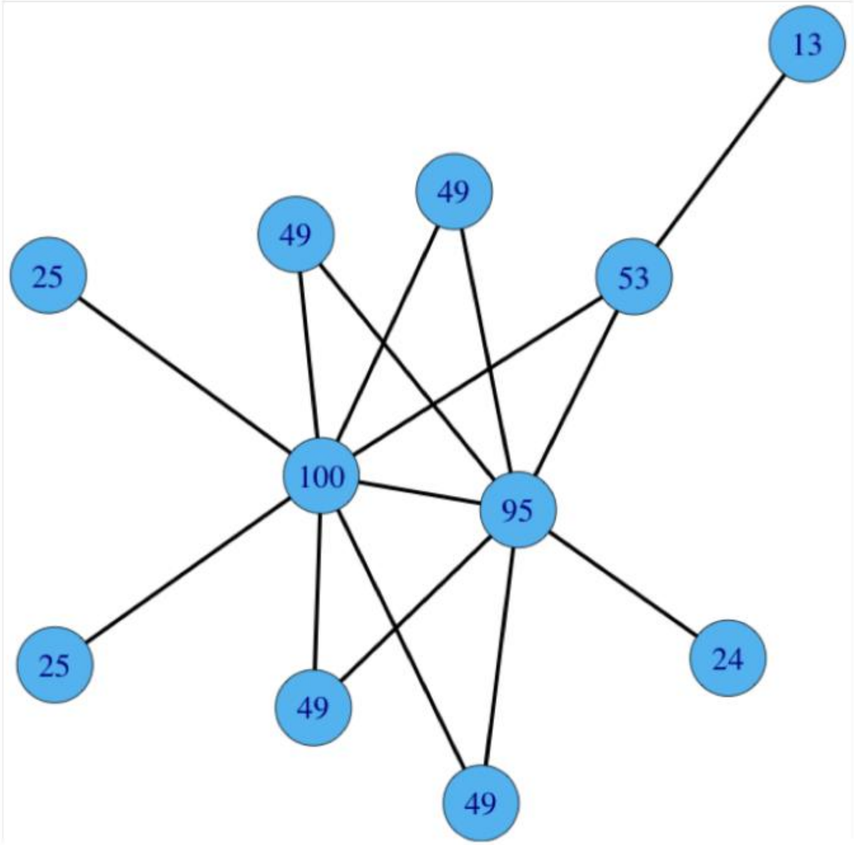
Iteration 2

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.34 \\ 0.68 \\ 0.51 \\ 0.17 \\ 0.34 \end{bmatrix} = \begin{bmatrix} 1.19 \\ 1.36 \\ 1.36 \\ 0.68 \\ 1.19 \end{bmatrix} \equiv \begin{bmatrix} 0.45 \\ 0.51 \\ 0.51 \\ 0.25 \\ 0.45 \end{bmatrix}$$

Iteration 3

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.45 \\ 0.51 \\ 0.51 \\ 0.25 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 1.02 \\ 1.66 \\ 1.41 \\ 0.51 \\ 1.02 \end{bmatrix} \equiv \begin{bmatrix} 0.38 \\ 0.62 \\ 0.53 \\ 0.19 \\ 0.38 \end{bmatrix} \quad \cdots \quad c = \begin{bmatrix} 1 \\ 1.41 \\ 1.27 \\ 0.52 \\ 1 \end{bmatrix}$$

Example



Eigenvalue centrality counts walks of length infinity

Eigencentrality

- **Spectral measures**
  - **Eigenvector Centrality**: Measure of the **influence** of a node in a network:

$$c_{\mathrm{eig}}(x) = \frac{1}{\lambda} \sum_{y \to x} c_{\mathrm{eig}}(y)$$

    - c_{eig} converges to the dominant eigenvector of adj. matrix $\boldsymbol{A}$
    - λ converges to the dominant eigenvalue of adj. matrix $\boldsymbol{A}$

  - Equivalently, eigencentrality is the eigenvector correspond to the dominant eigenvalue (λ) of $\boldsymbol{A}$

$$\boldsymbol{AX} = \lambda \boldsymbol{X}$$

Proof in next slide

Eigencentrality (Proof)

$$A_{ij} = 1 \quad \text{if there is an edge from node } j \text{ to node } i$$

$$\mathbf{c} = \begin{bmatrix} c_{\text{eig}}(1) \\ c_{\text{eig}}(2) \\ \vdots \\ c_{\text{eig}}(n) \end{bmatrix}$$

This is a vector that stores the centrality score for **every node** in the network.

$$c_{\text{eig}}(i) = \frac{1}{\lambda} \sum_{j=1}^{n} A_{ij}\, c_{\text{eig}}(j)$$

$$\Downarrow$$

$$\mathbf{c} = \frac{1}{\lambda} A\mathbf{c}$$

$$\Downarrow$$

$$A\mathbf{c} = \lambda \mathbf{c}$$

Why we choose the largest eigenvector as eigencentraliy? https://en.wikipedia.org/wiki/Perron%E2%80%93Frobenius_theorem

Problem with Eigencentrality

- **Spectral measures**
  - **Problem**: Graph should be **strongly connected**!

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Both blocks have eigenvalues:

$$\lambda = +1, -1$$

So the eigenvalues of the full matrix A are:

$$\lambda = 1, 1, -1, -1$$

There are two **independent eigenvectors** for $\lambda = 1$:

1. One is non-zero only in nodes 1–2 (e.g., $[1, 1, 0, 0]^T$)
2. The other is non-zero only in nodes 3–4 (e.g., $[0, 0, 1, 1]^T$)

The space of eigenvectors corresponding to λ=1 is 2-dimensional, so not unique.

Conclusion

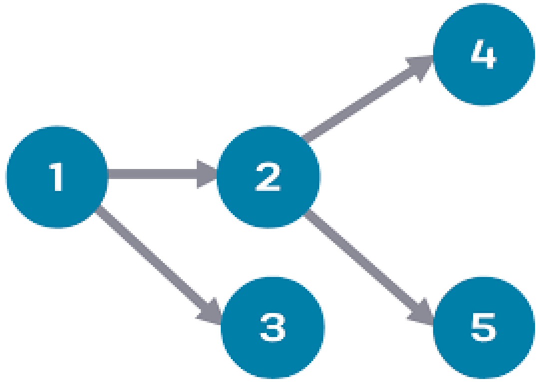
Eigencentrality solves a homogeneous equation:

$$Ac=\lambda c$$

- Finds the **dominant eigenvector**.
- Only works well when the graph is **strongly connected**.
- In **Directed Graph**, nodes without incoming links get zero centrality.

Katz's Index

- A major problem with eigenvector centrality arises when it deals with directed graphs
- Centrality only passes over outgoing edges and in special cases such as when a node is in a Directed Acyclic Graph (DAG) centrality becomes zero
  - The node can have many edge connected to it
- To resolve this problem we add bias term β to the centrality values for all nodes



Eigenvector Centrality

$$C_{\text{Katz}}(v_i) = \boxed{\alpha \sum_{j=1}^{n} (A_{j,i} C_{\text{Katz}}(v_j))} + \beta$$

Review

Matrix Induced P-Norm

Definition

$$\|A\|_p = \max_{\vec{x} \neq \vec{0}} \frac{\|A\vec{x}\|_p}{\|\vec{x}\|_p} = \max_{\|\vec{x}\|_p = 1} \|A\vec{x}\|_p$$

It's the **maximum amount** by which A stretches a vector of unit length.

Matrix Induced 2-Norm

Theorem:

$$\|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \sigma_1$$

SVD of A:

$$A = U\Sigma V^T \quad \text{with } U, V \text{ orthogonal}, \quad \Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$$

Let $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|_2 = 1$

$$\|A\mathbf{x}\|_2 = \|U\Sigma V^T\mathbf{x}\|_2 = \|\Sigma V^T\mathbf{x}\|_2$$

Let $\mathbf{y} = V^T\mathbf{x} \Rightarrow \|\mathbf{y}\|_2 = 1$

$$\|A\mathbf{x}\|_2 = \|\Sigma\mathbf{y}\|_2 = \sqrt{\sum_{i=1}^{r} \sigma_i^2 y_i^2}$$

Maximized when $y_1 = 1, y_{i>1} = 0$

$$\|A\mathbf{x}\|_2 \leq \sigma_1 = \sigma_{\max}(A)$$

So:

$$\|A\|_2 = \sigma_{\max}(A)$$

Matrix Induced 2-Norm

What happened for symmetric matrix?

$$||A||_2 = \max |\lambda_i|$$

Katz's Index

$$c_{\text{Katz}}(x) = \alpha \sum_{y \to x} c_{\text{Katz}}(y) + \beta$$

Normalization constant

$$\mathbf{c} = \alpha A \mathbf{c} + \beta \mathbf{1}$$

- The solution to Katz Centrality is:

$$\mathbf{c} = \beta (I - \alpha A)^{-1} \mathbf{1}$$ Expensive Inverting

Theorem: Neumann Series

$$\|\alpha A\|_2 = |\alpha| \cdot \|A\|_2 = |\alpha| \cdot \lambda_{\max} \quad \|\alpha A\|_2 < 1 \quad \Rightarrow \quad |\alpha| \cdot \lambda_{\max} < 1 \quad \Rightarrow \quad |\alpha| < \frac{1}{\lambda_{\max}}$$

$$\|\alpha A\| < 1 \quad \Longleftrightarrow \quad \alpha < \frac{1}{\lambda_{\max}}$$

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix.

If:

$$\|\alpha A\| < 1 \quad \text{(using some sub-multiplicative norm, e.g. 2-norm)}$$

then the inverse of $(I - \alpha A)$ exists and is given by the infinite series:

$$(I - \alpha A)^{-1} = \sum_{k=0}^{\infty} (\alpha A)^k \rightarrow (I - \alpha A)^{-1} = I + \alpha A + \alpha^2 A^2 + \alpha^3 A^3 + \cdots$$

This is called the **Neumann Series** (or matrix geometric series).

Katz's Index

$$c_{\text{katz}}(x) = \alpha \sum_{y \to x} c_{\text{Katz}}(y) + \beta$$

Normalization constant

$$\mathbf{c} = \alpha A \mathbf{c} + \beta \mathbf{1}$$

$$\mathbf{c} = \beta \sum_{k=0}^{\infty} (\alpha A)^k \mathbf{1} \qquad \alpha < \frac{1}{\lambda_{\max}}$$

- $A^1\mathbf{1}$: 1-step walks
- $A^2\mathbf{1}$: 2-step walks
- $A^k\mathbf{1}$: k-step walks
- Each one weighted by α^k

A weighted sum over all walks from all nodes to a given node — decayed with distance.

Katz's Index

- **Spectral measures**
  - **Katz's Index**: Measures **influence** by taking into account the **total number of walks** between a pair of nodes

$$c_{\text{katz}}(x) = \boldsymbol{\beta} \sum_{k=0}^{\infty} \sum_{x \to y} \alpha^k \boxed{\left(A^k\right)_{xy}}$$

Total number of walks of length k between x, y

  - α is an attenuation factor in range $\left(0, \frac{1}{\lambda}\right)$, where λ is the largest eigenvalue of A
  - β is to give some nodes more privilege
  - **Long paths are weighted less than short ones**

Katz's Index

- **Spectral measures**
  - **Katz's Index**: Measures **influence** by taking into account the **total number of walks** between a pair of nodes

$$c_{\text{katz}}(x) = \beta \sum_{k=0}^{\infty} \sum_{x \to y} \boxed{\alpha^k} \boxed{\left(A^k\right)_{xy}}$$

$\alpha < 1$: Long paths are weighted less

Total number of walks of length k between x, y

$A^0 = \mathbf{I}$

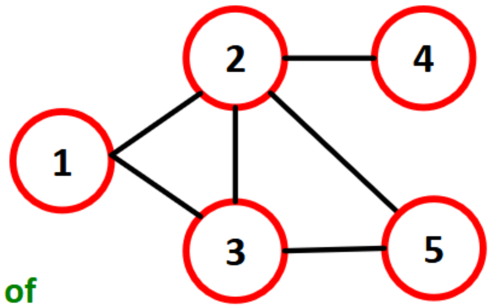
$$A^1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 4 & 2 & 0 & 1 \\ 1 & 2 & 3 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 6 & 5 & 1 & 2 \\ 6 & 4 & 6 & 4 & \boxed{6} \\ 5 & 6 & 4 & 2 & 5 \\ 1 & 4 & 2 & 0 & 1 \\ 2 & 6 & 5 & 1 & 2 \end{bmatrix}$$

Number of walks of length 3 between 2, 5

(2,1,3,5), (2,4,2,5), (2,3,2,5),
(2,1,2,5), (2,5,3,5), (2,5,2,5)



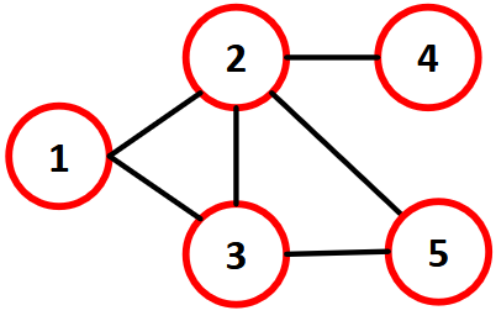
How to compute Katz's Index?

- **Spectral measures**
  - **Katz's Index**: Give each node a small amount of centrality for free

$$c_{\text{katz}}(x) = \alpha \sum_{y \to x} c_{\text{Katz}}(y) + \beta$$

Normalization constant

- **Power Iteration:**
  - Set $\boldsymbol{c}^{(0)} \leftarrow 1, k \leftarrow 1$
  - **1:** $\boldsymbol{c}^{(k)} \leftarrow \alpha \boldsymbol{A}\boldsymbol{c}^{(k-1)} + \beta \mathbf{1}$
  - **2:** If $\left\|\boldsymbol{c}^{(k)} - \boldsymbol{c}^{(k-1)}\right\| > \varepsilon$:
  - **3:** $\quad k \leftarrow k+1$, goto **1**



Instead of solving directly, we iterate this recurrence.

Katz's Index Behavior

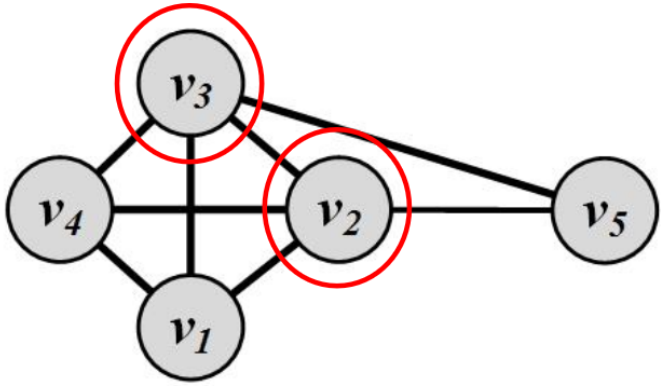
$$\mathbf{c} = \alpha A\mathbf{c} + \beta\mathbf{1} \qquad \mathbf{c} = \beta \sum_{k=0}^{\infty} (\alpha A)^k \mathbf{1}$$

α Value	Behavior of Katz Centrality	Interpretation
🟢 $\alpha = 0$	$\mathbf{c}_{\text{Katz}} = \beta\mathbf{1}$	All nodes get the **same score**
🟡 $0 < \alpha \ll 1$	$\mathbf{c}_{\text{Katz}} \approx \beta + \alpha \cdot A\mathbf{1}$	Like **in-degree centrality** (short paths)
🔵 Moderate α	Combines short & mid-length paths	Reflects **neighborhood influence**
🟣 $\alpha \to \frac{1}{\lambda_{\max}}^{-}$	Katz → **eigenvector centrality**	All walks included (infinite-length paths)
🔴 $\alpha = \frac{1}{\lambda_{\max}}$	❌ Katz **diverges** (matrix is not invertible)	The formula breaks; no meaningful output
❌ $\alpha > \frac{1}{\lambda_{\max}}$	❌ Series **does not converge**	Katz is **mathematically undefined**

Katz's Index

- **Spectral measures**
  - **Katz's Index**: Suitable for **directed acyclic graphs**
  - **How to choose α?**
    - For α **close to 0**, the contribution given by paths longer than one rapidly declines, and thus
      - Katz scores are mainly **influenced by short paths** (mostly in-degrees)
    - When the α **is large**, long paths are devalued smoothly, and
      - Katz scores are more **influenced the by topology** of the network
    - The measure diverges at $\alpha > \frac{1}{\lambda}$
    - The dominant eigenvector of A is the limit of Katz centrality as α approaches $1/\lambda$ from below

Example



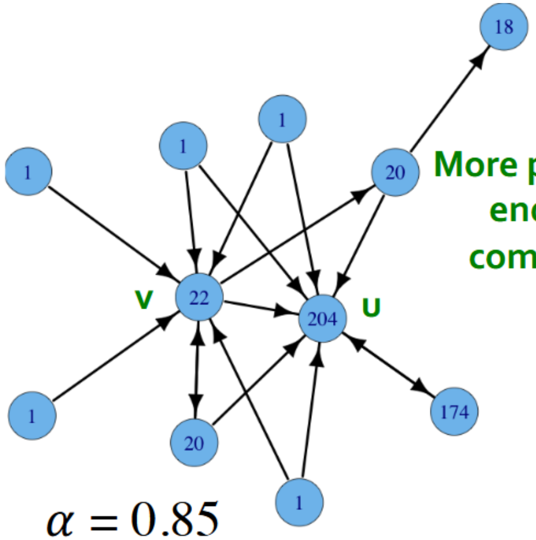
$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} = A^T$$

- The Eigenvalues are -1.68, -1.0, -1.0, 0.35, $\boxed{3.32}$
- We assume α=0.25 < $\boxed{1/3.32}$ and $\beta = 0.2$

$$\mathbf{C}_{Katz} = \beta(\mathbf{I} - \alpha A^T)^{-1} \cdot \mathbf{1} = \begin{bmatrix} 1.14 \\ 1.31 \\ 1.31 \\ 1.14 \\ 0.85 \end{bmatrix}$$

Most important nodes!

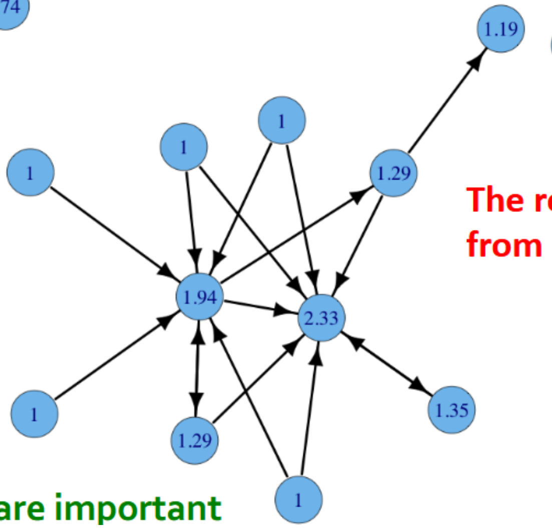
Example



More paths of length > 1 end up at node u, compared to node v

$\alpha = 0.85$
$\beta = 1$

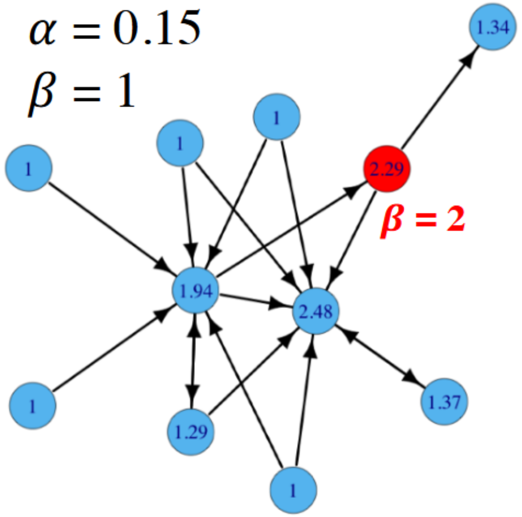
longer paths are important



$\alpha = 0.15$
$\beta = 1$

short paths are important

$\alpha = 0.15$
$\beta = 1$



The red node and all nodes reachable from it increase their centrality.

Conclusion

Kat'z solves the inhomogeneous equation:

$$\mathbf{c} = \alpha A\mathbf{c} + \beta\mathbf{1}$$

- Handles disconnected or acyclic graphs
- Gives non-zero centrality even to isolated nodes

Conclusion

- Problem:
  - In **directed graphs**, when a node has high centrality, it can pass that centrality **equally** to all out-neighbors.
    - This may overvalue nodes that are simply **linked by an important node** — even if they aren't important themselves.
- Solution:
  - We can divide the value of passed centrality by the number of outgoing links, i.e., out-degree of that node
  - Each connected neighbor gets a fraction of the source node's centrality

Introduction to PageRank

$$C_p(v_i) = \alpha \sum_{j=1}^{n} A_{j,i} \frac{C_p(v_j)}{d_j^{\text{out}}} + \beta$$

What if the degree is zero?

$$\begin{cases} d_j^{out} > 0 \\ D = diag(d_1^{\text{out}}, d_2^{\text{out}}, \ldots, d_n^{\text{out}}) \end{cases}$$

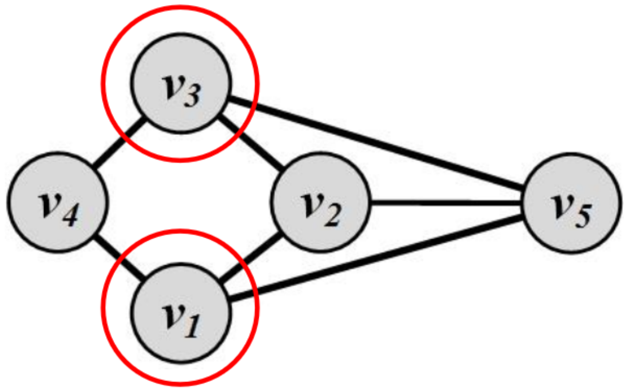
$$\mathbf{C}_p = \alpha A^T D^{-1} \mathbf{C}_p + \beta \mathbf{1}$$

$$\mathbf{C}_p = \beta (\mathbf{I} - \alpha A^T D^{-1})^{-1} \cdot \mathbf{1}$$

Similar to Katz Centrality, in practice, $\boldsymbol{\alpha} < \mathbf{1/\lambda}$, where λ is the largest eigenvalue of $A^T D^{-1}$. In undirected graphs, the largest eigenvalue of $A^T D^{-1}$ is $\boldsymbol{\lambda} = 1$; therefore, $\boldsymbol{\alpha} < \mathbf{1}$.

PageRank Example

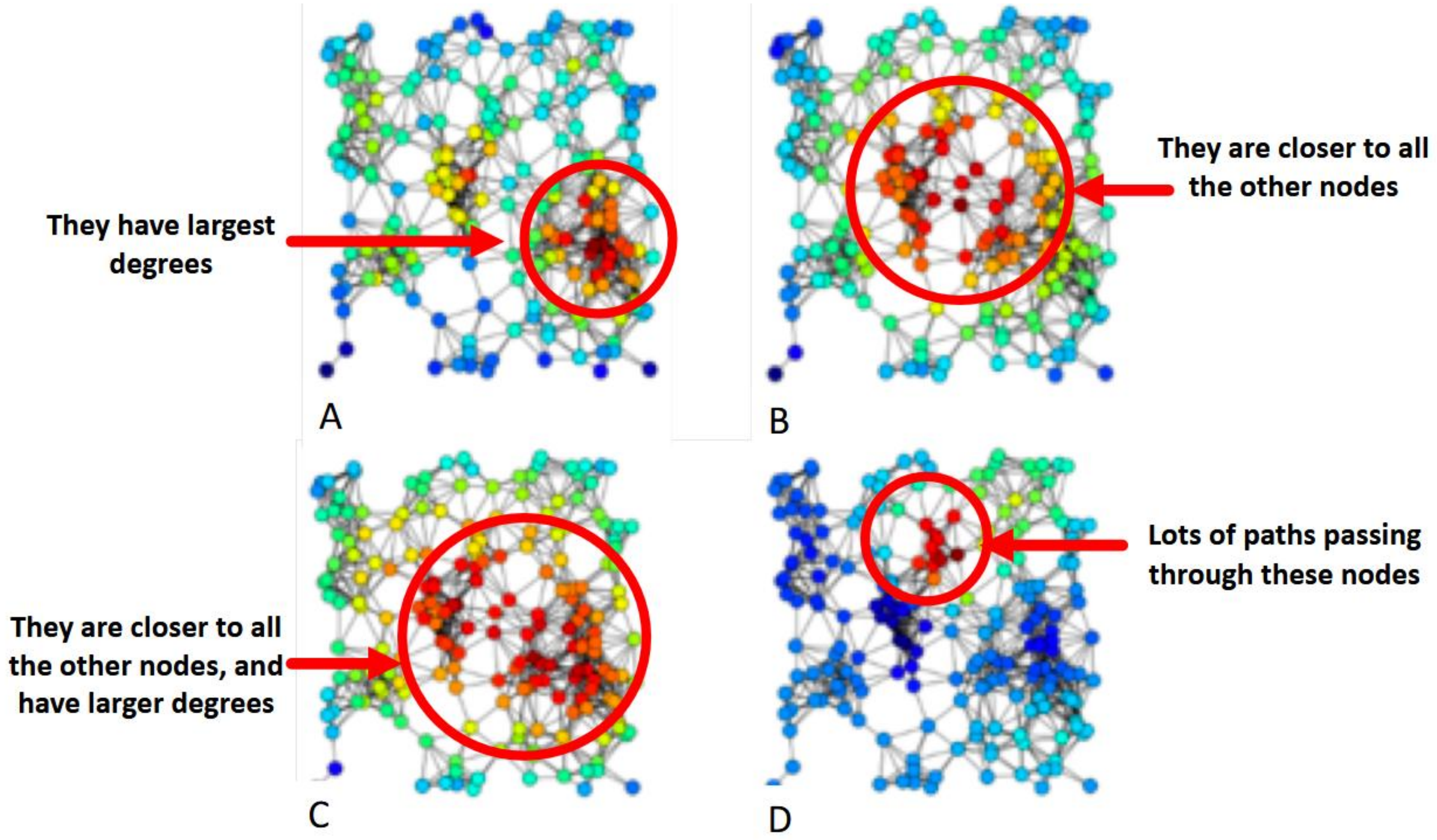
- We assume α=0.95 < 1 and and $\beta = 0.1$



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C}_p = \beta(\mathbf{I} - \alpha A^T D^{-1})^{-1} \cdot \mathbf{1} = \begin{bmatrix} 2.14 \\ 2.13 \\ 2.14 \\ 1.45 \\ 2.13 \end{bmatrix}$$

Example



Examples of A) Degree centrality, B) Closeness centrality, C) Harmonic Centrality and D) Katz centrality of the same graph.

An Interesting Comparison!

- Comparing three centrality values
  - Generally, the 3 centrality types will be positively correlated
  - When they are not (or low correlation), it usually reveals interesting information

	Low Degree	Low Closeness	Low Betweenness
High Degree		*Node is embedded in a community that is far from the rest of the network*	*Ego's connections are redundant - communication bypasses the node*
High Closeness	*Key node connected to important/active alters*		*Probably multiple paths in the network, ego is near many people, but so are many others*
High Betweenness	*Ego's few ties are crucial for network flow*	*Very rare! Ego monopolizes the ties from a small number of people to many others.*	

This slide is modified from a slide developed by James Moody

Any Question?